IRP SAFAIS-2019 "Spaces of Analytic Functions: Approximation, Interpolation, Sampling"

Extreme values of the Riemann zeta function and its argument

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Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad \Re(s) > 1.$$

Riemann zeta function

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Meromorphic continuation:

$$\zeta(s) = \frac{1}{s-1} + A(s), \qquad \Re(s) > 1,$$

$A(s)$ is an entire function.

Symmetry

Functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s).$$

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$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / x}$$

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$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx.$$

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Then we have

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1} \sum_{n=1}^\infty e^{-\pi n^2 x} dx, \quad \Re(s) > 1$$

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$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^1 + \int_1^\infty$$

$$= \frac{1}{s(s-1)} + \int_1^\infty (x^{-\frac{1}{2}s-\frac{1}{2}} + x^{\frac{1}{2}s-1}) \sum_{n=1}^\infty e^{-\pi n^2 x} dx.$$

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Therefore

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Trivial zeros:
$\zeta(s) = 0$, for $s = -2, -4, \ldots$.

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Trivial zeros:
$\zeta(s) = 0$, for $s = -2, -4, \dots$.
Explicit formula:

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = C + \frac{1}{s-1} - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \sum_{n\in\mathbb{N}}\left(\frac{1}{s+2n} - \frac{1}{2n}\right)$$

the Riemann-Weil explicit formula

Let h be a nice analytic function. Then

$$\sum_{\rho} h\left(\frac{\rho - 1/2}{i}\right) = h(i/2) + h(-i/2) + \frac{1}{\pi}\int_{-\infty}^{\infty} h(u)\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(1/2 + iu)du$$

$$- \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right),$$

where the sum is taken over all the nontrivial zeros of ζ.

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(B.-Radchenko-Seip, 2020) There are other explicit formulas related to ζ.

Riemann hypothesis

All nontrivial zeroes are on the critical line

Lindelöf hypothesis

For any $\epsilon > 0$

$$|\zeta(\frac{1}{2} + it)| = o(t^{\epsilon}), \qquad t \to \infty.$$

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Bourgain (2016):

$$|\zeta(\frac{1}{2} + it)| = O(t^{13/84+\epsilon}).$$

Lindelöf hypothesis

$$\zeta(\frac{1}{2}+it)=\sum_{n=1}^{N} n^{-1/2-it}-\frac{N^{1/2-it}}{1/2-it}+O(N^{-1/2}),\quad |t|<N.$$

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Should be a lot of cancelations!
Why for any large $a \in \mathbb{N}$ and $t < a^{1/\epsilon}$

$$a^{it}+(a+1)^{it}+\ldots+(2a)^{it}=o(a^{1/2+\epsilon})?$$

Asymmetry

Asymmetry

It seems that

$$\sum_{n=1}^{N} n^{-1/2-it}$$

attains "small values" on $[N/2, N]$, but doesn't attain "large values"!

Lower bounds

Montgomery; Balasubramanian and Ramachandra, 1977: $\exists$ large T with

$$|\zeta(1/2+iT)| \geq \exp\left(c\sqrt{\frac{\log T}{\log\log T}}\right).$$

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Theorem 1. (B, Seip; 2017)
$\exists$ large T with

$$|\zeta(1/2+iT)| \geq \exp\left(c\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\right),$$

where $c = 1/\sqrt{2} + o(1)$.

Further improvement

B., Seip (2017): $c = 1 + o(1)$
R. de la Bretèche, G. Tenenbaum (2018): $c = \sqrt{2} + o(1)$.

What is the truth?

Farmer–Gonek–Hughes (2007) have conjectured, by use of random matrix theory, that the right bound is

$$\exp\left((1/\sqrt{2}+o(1))\sqrt{\log T \log\log T}\right).$$

What is the truth?

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$$\exp\left((1/\sqrt{2}+o(1))\sqrt{\log T \log\log T}\right).$$

Example:

$$|\zeta(1/2+3.9246764...10^{31}i)| \approx 16244.$$

For this particular T

$$\exp\left(\sqrt{(1/2)\log T \log\log T}\right) \approx 264964,$$

$$\exp\left(\sqrt{\frac{2\log T \log\log\log T}{\log\log T}}\right) \approx 1128.$$

Sketch of the proof

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$$\int_{\mathbb{R}} F(t)dt = M_2$$

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Then for some $t \in [-T, T]$, $|\zeta(1/2 + it)| \gg M_1/M_2$.

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2. Choice of F

$$F = \Big|\sum_{m\in\mathcal{M}'} r(m)m^{it}\Big|^2 \Phi\Big(\frac{\log T}{T}t\Big),$$

where $\Phi(t) = e^{-t^2/2}$.

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$$M_1 = \frac{\sqrt{2\pi}\, T}{\log T} \sum_{m,n \in \mathcal{M}'} r(m) r(n) \Phi\Big(\frac{T}{\log T} \log \frac{m}{n}\Big) \ + \ \text{small terms.}$$

$$M_2 = \frac{\sqrt{2\pi}\, T}{\log T} \sum_{m,n \in \mathcal{M}'} \sum_{k \le T} \frac{r(m) r(n)}{\sqrt{k}} \Phi\Big(\frac{T}{\log T} \log \frac{km}{n}\Big) \ + \ \text{small terms.}$$

Sketch of the proof

3. Optimization problem

$|\mathcal{M}| = N \approx T^{1/2}$,

$$\sum_{m \in \mathcal{M}} f(m)^2 = 1$$

Maximize

$$\sum_{m,n \in \mathcal{M}, m=kn} \frac{f(n)f(m)}{\sqrt{k}}.$$

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Answer:

$$\exp\left(\sqrt{\frac{\log N \log\log\log N}{\log\log N}}\right).$$

How to choose $\mathcal{M}$ and f?

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f is a multiplicative function, that is $f(mn) = f(n)f(m)$ for $(m, n) = 1$ supported on square-free numbers,

$$f(p) := \sqrt{\frac{\log N \log_2 N}{\log_3 N}} \frac{1}{\sqrt{p} \log p},$$

$$p \le \log N \exp((\log_2 N)^{1-o(1)}).$$

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$\mathcal{M}$ is the set where the "mass" of f lives.
Approach works for Dirichlet polynomials with positive coefficients.

Further improvement by convolution formula

Theorem 2.(B, Seip; 2017)
$\exists$ large T with

$$|\zeta(1/2+iT)| \geq \exp\left(c\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\right),$$

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Further improvement by convolution formula

Therefore for a "nice" K we have

$$\int_{-\infty}^{\infty} \zeta(\sigma + i(t+u))K(u)du = \sum_{n=1}^{\infty} \widehat{K}(\frac{\log n}{2\pi})n^{-\sigma-it} - 2\pi K(t - i(1-\sigma)).$$

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Choose K such a way that reminder term will be small!

$S(T)$

$$S(t) := \frac{1}{\pi}\Im \log \zeta(1/2 + it)$$

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Riemann–von Mangoldt formula:

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right),$$

where as usual $N(t)$ is the number of zeros $\beta + i\gamma$ of $\zeta(s)$ for which $0 < \gamma < t$.

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Theorem 3.(B, Seip, 2017) On the Riemann hypothesis there are arbitrarily large T with

$$|S(T)| \geq c\sqrt{\frac{\log T \log \log \log T}{\log \log T}}$$

GCD sums

What is the maximum of

$$\sum_{m,n\in\mathcal{M}} c_m c_n \frac{(m,n)}{\sqrt{mn}},$$

where

$$\sum_{m\in\mathcal{M}} c_n^2 = 1?$$

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Or

$$\frac{1}{|\mathcal{M}|}\sum_{m,n\in\mathcal{M}} \frac{(m,n)}{\sqrt{mn}}?$$

They are almost the same! Reason: (m,n) is a certain inner product.

GCD sums

Example: If $\mathcal{M}$ are all divisors of $p_1 \dots p_\ell$ then

$$\frac{1}{|\mathcal{M}|} \sum_{m,n \in \mathcal{M}} \frac{(m,n)}{\sqrt{mn}} = \prod_{j=1}^{\ell} (1 + p_j^{-1/2})$$

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- It is enough to consider square free numbers
- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.

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How to prove upper bounds?

- It is enough to consider square free numbers
- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.
- Divisor closed extremal sets $\mathcal{M}$ enjoy the following completeness property: If $n \in \mathcal{M}$, $p|n$, $p' < p$, then either $p'|n$ or $p'n/p \in \mathcal{M}$.

GCD sums

Combining the last with Aistleitner–Berkes–Seip arguments we obtain.

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Combining the last with Aistleitner–Berkes–Seip arguments we obtain.
Theorem 4.(B, Seip, 2015)

$$\frac{1}{N}\sup_{1\le n_1\dots<n_N}\sum_{k,\ell=1}^{N}\frac{(n_k,n_\ell)}{\sqrt{n_k n_\ell}}\approx\exp\left(A\sqrt{\frac{\log N\log\log\log N}{\log\log N}}\right),$$

where $1\le A<7$.
Other tools: Bohr correspondence, multiplicative functions, Chauchy–Shwarz inequality

Questions

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- How to improve the lower bound?

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- How to improve the upper bound?

THANK YOU!