Class 16: Extreme values and conditional bounds for $\zeta(s)$

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28-October-2021

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5. Conjecture: $E(T) \ll T^{1/4+\varepsilon}$.

Using the functional equation, we proved that

$$\zeta(s) = O(|t|^{3/2+\delta}), \quad \text{for } \sigma \geq -\delta.$$

Therefore, for any semiplane $\sigma \geq \sigma_0$ we have

$$|\zeta(s)| = O(|t|^k),$$

for some k depending on σ_0. This implies that $\zeta(s)$ is a function of finite order in the sense of the theory of Dirichlet series.

For any σ we define $\mu(\sigma)$ as the infimum of the values ξ such that

$$\zeta(\sigma + it) = O\big(|t|^{\xi}\big).$$

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2. Using the approximation formula: $\mu(\frac{1}{2}) \leq \frac{1}{2}$.

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5. $\mu(\sigma) = \frac{1}{2} - \sigma$, *for* $\sigma \leq 0$.

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5. $\mu(\sigma) = \frac{1}{2} - \sigma$, *for* $\sigma \leq 0$.
6. μ *is a decreasing function.*

In particular, using the fact that $\mu(0) = \frac{1}{2}$ and $\mu(1) = 0$, it follows for $0 < \sigma < 1$:

$$\mu(\sigma) \leq \frac{1}{2} - \frac{\sigma}{2}.$$

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This implies that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| = O\left(|t|^{\frac{1}{4}+\varepsilon}\right).$$

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This is called: **Convexity bound**

Lindelöf hypothesis-1908

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$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = O\big(|t|^{\varepsilon}\big).$$

$\mu(1/2) \leq$	$\mu(1/2) \leq$	Author	
1/4	0.25	Lindelöf (1908)	Convexity bound
1/6	0.1667	Hardy, Littlewood & ?	
163/988	0.1650	Walfisz (1924)	
27/164	0.1647	Titchmarsh (1932)	
229/1392	0.164512	Phillips (1933)	
	0.164511	Rankin (1955)	
19/116	0.1638	Titchmarsh (1942)	
15/92	0.1631	Min (1949)	
6/37	0.16217	Haneke (1962)	
173/1067	0.16214	Kolesnik (1973)	
35/216	0.16204	Kolesnik (1982)	
139/858	0.16201	Kolesnik (1985)	
32/205	0.1561	Huxley (2002, 2005)	
53/342	0.1550	Bourgain (2017)	
13/84	0.1548	Bourgain (2017)	

Then, for all t sufficiently large:

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(\left(\frac{13}{84}+\epsilon\right)\log t\right).$$

Omega results

(1) Titchmarsh (1928): for any $\varepsilon > 0$ there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c(\epsilon)(\log t)^{\frac{1}{2}-\epsilon}\right).$$

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(2) Levinson (1972): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c\,\frac{(\log t)^{\frac{1}{2}}}{\log\log t}\right).$$

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(3) Balasubramanian and Ramachandra (1977): for some constant $c > 0$ there are infinitely many t sufficiently large such that

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(4) Montgomery (1977): assuming RH, there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(\frac{1}{20}\,\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Omega results

(5) Soundararajan (2008): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left((1+o(1))\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

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Omega results

(6) Bondarenko and Seip (2017): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\frac{(\log t)^{\frac{1}{2}}(\log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

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(8) R. de la Bretèche and Tenenbaum (2018): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left((\sqrt{2}+o(1))\frac{(\log t)^{\frac{1}{2}}(\log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Idea of the proof

We will use the classical resonance method of Soundararajan in the version of Bondarenko and Seip. We find a certain Dirichlet polynomial which "resonates" with the $|\zeta(\frac{1}{2}+it)|$, i.e. that pick large values of zeta. The resonator will be $|\mathcal{R}(t)|^2$, where

$$\mathcal{R}(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it} = \sum_{m \leq N} r(m) m^{-it},$$

and $\mathcal{M}'$ is a suitable finite set of integers and $r(m)$ is an arithmetic function.

Soundararajan's version

Let $\varphi(t)$ be a smooth function compactly supported in $[1,2]$, such that $0 \le \varphi(t) \le 1$ and $\varphi(t) = 1$, for $t \in (5/4, 7/4)$. He computed

$$M_1(\mathcal{R}, T) = \int_{-\infty}^{\infty} |\mathcal{R}(t)|^2 \varphi\left(\frac{t}{T}\right) \mathrm{d}t,$$

and

$$M_2(\mathcal{R}, T) = \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2} + it) |\mathcal{R}(t)|^2 \varphi\left(\frac{t}{T}\right) \mathrm{d}t.$$

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Then

$$\frac{|M_2(\mathcal{R}, T)|}{M_1(\mathcal{R}, T)} \leq \max_{t \in [T, 2T]} |\zeta(\tfrac{1}{2} + it)|.$$

Considering $N \leq T^{1-\varepsilon}$:

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\begin{aligned}
M_1(\mathcal{R}, T) &= \int_{-\infty}^{\infty} |\mathcal{R}(t)|^2 \varphi\left(\frac{t}{T}\right) \mathrm{d}t \\
&= T \sum_{m,n \leq N} r(m)\, \overline{r(n)}\, \widehat{\varphi}\left(T \log \frac{m}{n}\right) \\
&= T\, \widehat{\varphi}(0) \sum_{m \leq N} |r(m)|^2 + \text{small terms.}
\end{aligned}
$$

From the approximation formula  we have for $T \leq t \leq 2T$:

$$\zeta\left(\frac{1}{2}+it\right) = \sum_{k \leq T} \frac{1}{k^{\frac{1}{2}+it}} + O(T^{-\frac{1}{2}}).$$

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Therefore, considering $N \leq T^{1-\varepsilon}$:

$$\begin{aligned} M_2(\mathcal{R}, T) &= \int_{-\infty}^{\infty} \zeta(\tfrac{1}{2}+it)|\mathcal{R}(t)|^2 \varphi\left(\frac{t}{T}\right)\mathrm{d}t \\ &= T \sum_{m,n\leq N} \sum_{k\leq T} \frac{r(m)\,\overline{r(n)}}{k^{\frac{1}{2}}}\,\widehat{\varphi}\left(T \log \frac{mk}{n}\right) + \text{small terms.} \\ &= T\,\widehat{\varphi}(0) \sum_{mk=n\leq N} \frac{r(m)\,\overline{r(n)}}{k^{\frac{1}{2}}} + \text{small terms.} \end{aligned}$$

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Soundararajan proved that:

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and using $N = T^{1-\varepsilon}$ we obtain the desired result.

Bondarenko and Seip's version

Inspired in GCD-sums, they constructed a certain

$$\mathcal{R}(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it},$$

where $|\mathcal{M}'| \leq T^{\kappa}$ for $\kappa \leq 1/2$ and let $\Phi(t) = e^{-\frac{t^2}{2}}$.

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$$M_1(\mathcal{R}, T) = \int_{\sqrt{T} \leq |t| \leq T} |\mathcal{R}(t)|^2 \, \Phi\left(\frac{\log T}{T} t \right) \mathrm{d}t,$$

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$$\frac{|M_2(\mathcal{R}, T)|}{M_1(\mathcal{R}, T)} \leq \max_{t \in [\sqrt{T}, T]} |\zeta(\tfrac{1}{2} + it)|.$$

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M_1(\mathcal{R}, T) &= \int_{\sqrt{T}\le|t|\le T} |\mathcal{R}(t)|^2\, \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t \\
&\le \int_{-\infty}^{\infty} |\mathcal{R}(t)|^2\, \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t \\
&= \frac{T}{\log T} \sum_{m,n\in\mathcal{M}'} r(m)\, r(n)\, \widehat{\Phi}\left(\frac{T}{\log T} \log \frac{m}{n}\right) \\
&= \frac{\sqrt{2\pi}\, T}{\log T} \sum_{m,n\in\mathcal{M}'} r(m)\, r(n)\, \Phi\left(\frac{T}{\log T} \log \frac{m}{n}\right) \\
&\ll_{\mathcal{R}} T(\log T)^3.
\end{aligned}$$

From the approximation formula  we have for $T \leq t \leq 2T$:

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&\int_{-\infty}^{\infty}\left(\sum_{k\le T}\frac{1}{k^{\frac12+it}}\right)|\mathcal{R}(t)|^2\Phi\left(\frac{t}{T}\right)\mathrm{d}t \\
&= \frac{\sqrt{2\pi}T}{\log T}\sum_{m,n\in\mathcal{M}'}\sum_{k\le T}\frac{r(m)\,r(n)}{k^{\frac12}}\Phi\left(\frac{T}{\log T}\log\frac{km}{n}\right) \\
&\ge \frac{\sqrt{2\pi}T}{\log T}\sum_{m,n\in\mathcal{M}'}\sum_{k\le T,\,k:\text{special}}\frac{r(m)\,r(n)}{k^{\frac12}}\Phi\left(\frac{T}{\log T}\log\frac{km}{n}\right) \\
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\end{aligned}$$

where $N = |\mathcal{M}'|$ and

$$A_N = \prod_{p:\,\text{certain primes}}\frac{1+f(p)^2+f(p)p^{-1/2}}{1+f(p)^2}.$$

Bondarenko and Seip established that

$$A_N \geq \exp\left((\gamma + o(1))\frac{(\log N)^{\frac{1}{2}}(\log\log\log N)^{\frac{1}{2}}}{(\log\log N)^{\frac{1}{2}}}\right),$$

where $\gamma = 1 - \varepsilon$.

Considering $N = [T^\kappa]$ with $\kappa \le 1/2$:

$$\begin{aligned} M_2(\mathcal{R}, T) &= \int_{\sqrt{T} \le |t| \le T} \zeta(\tfrac{1}{2} + it) |\mathcal{R}(t)|^2 \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t \\ &\gg_{\mathcal{R}} \frac{T}{\log T} A_N \\ &\gg_{\mathcal{R}} \frac{T}{\log T} \exp\left((\gamma + o(1)) \frac{(\kappa \log T)^{\frac{1}{2}} (\log\log\log T)^{\frac{1}{2}}}{(\log\log T)^{\frac{1}{2}}} \right). \end{aligned}$$

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$$|M_2(\mathcal{R}, T)| \gg_{\mathcal{R}} \frac{T}{\log T} \exp\left((\gamma + o(1)) \frac{(\kappa \log T)^{\frac{1}{2}} (\log\log\log T)^{\frac{1}{2}}}{(\log\log T)^{\frac{1}{2}}}\right),$$

with $\kappa \leq 1/2$, and $\gamma = 1 - \varepsilon$. Using the inequality

$$\max_{t \in [\sqrt{T}, T]} |\zeta(\tfrac{1}{2} + it)| \geq \frac{|M_2(\mathcal{R}, T)|}{M_1(\mathcal{R}, T)},$$

we get the desired result.

What happens if we assume Riemann Hypothesis?

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Unconditionally, for all t sufficiently large:

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(\left(\frac{13}{84}+\epsilon\right)\log t\right).$$

Littlewood's result

A classical result of Littlewood (1924) states that, under the Riemann hypothesis, there is $C > 0$ such that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll \exp\left(C\frac{\log t}{\log\log t}\right).$$

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for t sufficiently large. The order of magnitude has not been improved over the last ninety years, and the efforts have hence been concentrated in optimizing the values of the implicit constants.

Assuming the Riemann hypothesis, we have

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(5) Carneiro, Chirre and Milinovich (2017) : Other proof.

Idea of the proof

The proof of these results consists of the following steps:

- Representation lemma: to express the desired object as sums over the zeros of $\zeta(s)$.
- Explicit formulas: the tools to evaluate such sums
- Harmonic analysis tools: find appropriate majorants/minorants to plug in.
- Evaluation of the terms.

Lemma (Representation lemma)

Assume the Riemann hypothesis. We define the function $f : \mathbb{R}^* \to \mathbb{R}$ *by*

$$f(x) = \log\left(\frac{4+x^2}{x^2}\right).$$

Then, for $t > 0$ *sufficiently large we have*

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

The sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ *of* $\zeta(s)$.

It's time to call our Guinand-Weil explicit formula!