Class 17: Conditional bounds for $\zeta(s)$

Andrés Chirre
Norwegian University of Science and Technology - NTNU

01-November-2021

Omega results

(1) Titchmarsh (1928): for any $\varepsilon > 0$ there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c(\epsilon)(\log t)^{\frac{1}{2}-\epsilon}\right).$$

Omega results

(1) Titchmarsh (1928): for any $\varepsilon > 0$ there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c(\epsilon)(\log t)^{\frac{1}{2}-\epsilon}\right).$$

(2) Levinson (1972): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c\,\frac{(\log t)^{\frac{1}{2}}}{\log\log t}\right).$$

Omega results

(3) Balasubramanian and Ramachandra (1977): for some constant $c > 0$ there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \gg \exp\left(c\,\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Omega results

(3) Balasubramanian and Ramachandra (1977): for some constant $c > 0$ there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c\,\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

(4) Montgomery (1977): assuming RH, there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(\frac{1}{20}\,\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Omega results

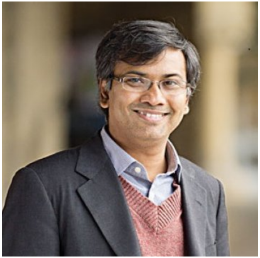
(5) Soundararajan (2008): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left((1+o(1))\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

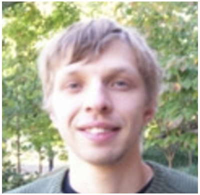
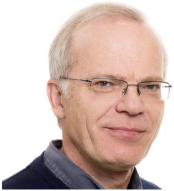
Omega results

(5) Soundararajan (2008): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left((1+o(1))\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$



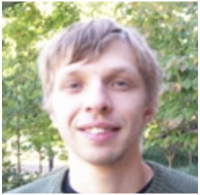
Omega results

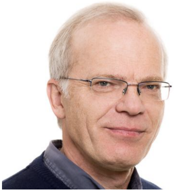
 

(6) Bondarenko and Seip (2017): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\frac{(\log t)^{\frac{1}{2}}(\log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Omega results





(7) Bondarenko and Seip (2017): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left((1+o(1))\frac{(\log t)^{\frac{1}{2}}(\log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Omega results

(8) R. de la Bretèche and Tenenbaum (2018): there are infinitely many t sufficiently large such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left((\sqrt{2}+o(1))\frac{(\log t)^{\frac{1}{2}}(\log\log\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right).$$

Idea of the proof

We will use the classical resonance method of Soundararajan in the version of Bondarenko and Seip. We find a certain Dirichlet polynomial which "resonates" with the $|\zeta(\frac{1}{2}+it)|$, i.e. that pick large values of zeta. The resonator will be $|\mathcal{R}(t)|^2$, where

$$\mathcal{R}(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it} = \sum_{m \leq N} r(m) m^{-it},$$

and $\mathcal{M}'$ is a suitable finite set of integers and $r(m)$ is an arithmetic function.

Bondarenko and Seip's version

Inspired in GCD-sums, they constructed a certain

$$\mathcal{R}(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it},$$

where $|\mathcal{M}'| \leq T^{\kappa}$ for $\kappa \leq 1/2$ and let $\Phi(t) = e^{-\frac{t^2}{2}}$.

Bondarenko and Seip's version

Inspired in GCD-sums, they constructed a certain

$$\mathcal{R}(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it},$$

where $|\mathcal{M}'| \leq T^{\kappa}$ for $\kappa \leq 1/2$ and let $\Phi(t) = e^{-\frac{t^2}{2}}$.

$$M_1(\mathcal{R}, T) = \int_{\sqrt{T} \leq |t| \leq T} |\mathcal{R}(t)|^2 \, \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t,$$

and

$$M_2(\mathcal{R}, T) = \int_{\sqrt{T} \leq |t| \leq T} \zeta(\tfrac{1}{2} + it) |\, \mathcal{R}(t)|^2 \, \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t.$$

Bondarenko and Seip's version

Inspired in GCD-sums, they constructed a certain

$$\mathcal{R}(t) = \sum_{m \in \mathcal{M}'} r(m) m^{-it},$$

where $|\mathcal{M}'| \leq T^{\kappa}$ for $\kappa \leq 1/2$ and let $\Phi(t) = e^{-\frac{t^2}{2}}$.

$$M_1(\mathcal{R}, T) = \int_{\sqrt{T} \leq |t| \leq T} |\mathcal{R}(t)|^2 \, \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t,$$

and

$$M_2(\mathcal{R}, T) = \int_{\sqrt{T} \leq |t| \leq T} \zeta(\tfrac{1}{2} + it) |\, \mathcal{R}(t)|^2 \, \Phi\left(\frac{\log T}{T} t\right) \mathrm{d}t.$$

Then

$$\frac{|M_2(\mathcal{R}, T)|}{M_1(\mathcal{R}, T)} \leq \max_{t \in [\sqrt{T}, T]} |\zeta(\tfrac{1}{2} + it)|.$$

What happens if we assume Riemann Hypothesis?

What happens if we assume Riemann Hypothesis?

Unconditionally, for all t sufficiently large:

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(\left(\frac{13}{84}+\epsilon\right)\log t\right).$$

Result of Littlewood

A classical result of Littlewood (1924) states that, under the Riemann hypothesis, there is $C > 0$ such that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll \exp\left(C \frac{\log t}{\log \log t}\right).$$

for t sufficiently large.

Result of Littlewood

A classical result of Littlewood (1924) states that, under the Riemann hypothesis, there is $C > 0$ such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(C\frac{\log t}{\log\log t}\right).$$

for t sufficiently large. The order of magnitude has not been improved over the last ninety years, and the efforts have hence been concentrated in optimizing the values of the implicit constants.

Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \le \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(1) Ramachandra and Sankaranarayanan (1993) : $C = 0.466$.

Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(1) Ramachandra and Sankaranarayanan (1993) : $C = 0.466$.
(2) Soundararajan (2009) : $C = 0.373$.

Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(1) Ramachandra and Sankaranarayanan (1993) : $C = 0.466$.
(2) Soundararajan (2009) : $C = 0.373$.
(3) Chandee and Soundararajan (2011) : $C = \frac{\ln(2)}{2} \approx 0.347$.
In this case $o(1) = \frac{\log\log\log t}{\log\log t}$.

Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

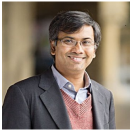
(4) Carneiro and Chandee (2011) : $C = \frac{\log 2}{2} \approx 0.347$.
In this case $o(1) = \frac{1}{\log\log t}$.

Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(4) Carneiro and Chandee (2011) : $C = \frac{\log 2}{2} \approx 0.347$.
In this case $o(1) = \frac{1}{\log\log t}$.

(5) Carneiro, Chirre and Milinovich (2017) : Other proof.

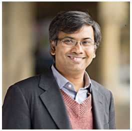


Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(4) Carneiro and Chandee (2011) : $C = \frac{\log 2}{2} \approx 0.347$.
In this case $o(1) = \frac{1}{\log\log t}$.

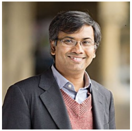
(5) Carneiro, Chirre and Milinovich (2017) : Other proof.



Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(4) Carneiro and Chandee (2011) : $C = \frac{\log 2}{2} \approx 0.347$.
In this case $o(1) = \frac{1}{\log\log t}$.
(5) Carneiro, Chirre and Milinovich (2017) : Other proof.

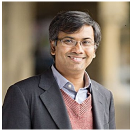


Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(4) Carneiro and Chandee (2011) : $C = \frac{\log 2}{2} \approx 0.347$.
In this case $o(1) = \frac{1}{\log\log t}$.

(5) Carneiro, Chirre and Milinovich (2017) : Other proof.

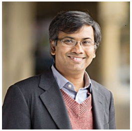


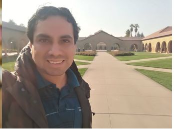
Assuming the Riemann hypothesis, we have

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left((C+o(1))\frac{\log t}{\log\log t}\right).$$

(4) Carneiro and Chandee (2011) : $C=\frac{\log 2}{2}\approx 0.347$.
In this case $o(1)=\frac{1}{\log\log t}$.

(5) Carneiro, Chirre and Milinovich (2017) : Other proof.





Lemma (Representation lemma)

Assume the Riemann hypothesis. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *be the function*

$$f(x) = \log\left(\frac{4 + x^2}{x^2}\right).$$

Then, for $t > 0$ *sufficiently large we have*

$$\log\left|\zeta\left(\frac{1}{2} + it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t - \gamma) + O(1).$$

The sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ *of* $\zeta(s)$.

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \le \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\, s| \to \infty$. Then

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi} \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Idea of the proof

The proof of these results consists of the following steps:

- Representation lemma: to express the desired object as sums over the zeros of $\zeta(s)$.
- Explicit formulas: the tools to evaluate such sums
- Harmonic analysis tools: find appropriate majorants/minorants to plug in.
- Evaluation of the terms.

Connection to Fourier analysis

- We have written our object in consideration as

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

- From the explicit formula it would be very nice if we could find an special function m such that
  - $m \leq f$.

Connection to Fourier analysis

- We have written our object in consideration as

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

- From the explicit formula it would be very nice if we could find an special function m such that
  - $m \leq f$.
  - $\widehat{m}$ has compact supports, say $[-\delta, \delta]$.

Connection to Fourier analysis

- We have written our object in consideration as

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

- From the explicit formula it would be very nice if we could find an special function m such that
  - $m \leq f$.
  - $\widehat{m}$ has compact supports, say $[-\delta, \delta]$.
  - We need to minimize

$$\int_{-\infty}^{\infty} f(x) - m(x)\, dx.$$

Connection to Fourier analysis

- We have written our object in consideration as

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

- From the explicit formula it would be very nice if we could find an special function m such that
  - $m \leq f$.
  - $\widehat{m}$ has compact supports, say $[-\delta,\delta]$.
  - We need to minimize

$$\int_{-\infty}^{\infty} f(x) - m(x)\, dx.$$

This is Beurling-Selberg's problem!!!

Developments of Beurling-Selberg's problem

Function	Optimal entire approximations
sgn(x)	Beurling 30's
$\chi_{[a,b]}(x)$	Selberg 50's and Logan 80's
$e^{-\lambda\|x\|}$	Graham-Vaaler '81
Even functions (e.g. $\log\|x\|$)	Carneiro-Vaaler '09
Even functions (e.g. $e^{-\lambda x^2}$) (Gaussian subordination)	Carneiro-Littmann-Vaaler '10
Odd functions (e.g. $sgn(x)e^{-\lambda x^2}$) (odd Gaussian subordination)	Carneiro-Vaaler '11

Theorem (Carneiro and Vaaler (TAMS))

Let ν be a measure defined on the Borel sets of $(0, \infty)$ such that

$$0 < \int_0^\infty \frac{\lambda}{\lambda^2 + 1} \, d\nu(\lambda) < \infty.$$

Define the function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$f(x) = \int_0^\infty \left\{ e^{-\lambda|x|} - e^{-\lambda} \right\} d\nu(\lambda),$$

where $f(0)$ may take the value ∞.

Theorem (Carneiro and Vaaler (TAMS))

Let ν *be a measure defined on the Borel sets of* $(0,\infty)$ *such that*

$$0 < \int_0^\infty \frac{\lambda}{\lambda^2+1}\, d\nu(\lambda) < \infty.$$

Define the function $f:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ *given by*

$$f(x) = \int_0^\infty \left\{ e^{-\lambda|x|} - e^{-\lambda} \right\} d\nu(\lambda),$$

where $f(0)$ *may take the value* ∞. *Then, there exists a unique extremal minorant* $G(z)$ *of exponential type* 2π *for* f. *The function* $G(x)$ *interpolates the values of* $f(x)$ *at* $\mathbb{Z}+\frac{1}{2}$.

$$G(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{f(n-\frac{1}{2})}{(z-n+\frac{1}{2})^2} + \sum_{n\in\mathbb{Z}} \frac{f'(n-\frac{1}{2})}{(z-n+\frac{1}{2})} \right\},$$

Let $\Delta > 0$, and consider the measure

$$d\nu_{\Delta}(\lambda) := \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda.$$

Let $\Delta > 0$, and consider the measure

$$d\nu_\Delta(\lambda) := \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda.$$

Then

$$\begin{aligned}\int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda \\ &= \log\left(\frac{4\Delta^2+x^2}{x^2}\right) - \log(4\Delta^2+1) \\ &= f_\Delta(x),\end{aligned}$$

Let $\Delta > 0$, and consider the measure

$$d\nu_\Delta(\lambda) := \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda.$$

Then

$$\begin{aligned}\int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda \\ &= \log\left(\frac{4\Delta^2 + x^2}{x^2}\right) - \log(4\Delta^2+1) \\ &= f_\Delta(x),\end{aligned}$$

where

$$f_\Delta(x) = f\left(\frac{x}{\Delta}\right) - f\left(\frac{1}{\Delta}\right).$$

Let $G_\Delta(z)$ be the minorant of exponential type 2π for f_Δ.

Let $G_\Delta(z)$ be the minorant of exponential type 2π for f_Δ. Then:

$$G_\Delta(x) \leq f_\Delta(x) = f\left(\frac{x}{\Delta}\right) - f\left(\frac{1}{\Delta}\right).$$

Let $G_\Delta(z)$ be the minorant of exponential type 2π for f_Δ. Then:

$$G_\Delta(x) \leq f_\Delta(x) = f\left(\frac{x}{\Delta}\right) - f\left(\frac{1}{\Delta}\right).$$

Define

$$H_\Delta(x) := G_\Delta(x) + f\left(\frac{1}{\Delta}\right) \leq f\left(\frac{x}{\Delta}\right).$$

Let $G_\Delta(z)$ be the minorant of exponential type 2π for f_Δ. Then:

$$G_\Delta(x) \leq f_\Delta(x) = f\left(\frac{x}{\Delta}\right) - f\left(\frac{1}{\Delta}\right).$$

Define

$$H_\Delta(x) := G_\Delta(x) + f\left(\frac{1}{\Delta}\right) \leq f\left(\frac{x}{\Delta}\right).$$

Finally, we define

$$m_\Delta(x) = H_\Delta(\Delta x) \leq f(x),$$

where $m_\Delta(z)$ is an entire function of exponential type $2\pi\Delta$.

Proposition (Chandee and Vaaler)

Let $\Delta \geq 1$. Then $m_\Delta : \mathbb{C} \to \mathbb{C}$ is an even entire function such that:

Proposition (Chandee and Vaaler)

Let $\Delta \geq 1$. Then $m_\Delta : \mathbb{C} \to \mathbb{C}$ is an even entire function such that:

$$(I) \quad \frac{-C}{1+x^2} \leq m_\Delta(x) \leq f(x), \text{ for some } C > 0 \text{ and for all } x \in \mathbb{R}.$$

Proposition (Chandee and Vaaler)

Let $\Delta \geq 1$. *Then* $m_\Delta : \mathbb{C} \to \mathbb{C}$ *is an even entire function such that:*

$$(I) \quad \frac{-C}{1+x^2} \leq m_\Delta(x) \leq f(x), \text{ for some } C > 0 \text{ and for all } x \in \mathbb{R}.$$

$$(II) \quad m_\Delta(z) \ll \frac{\Delta^2}{1+\Delta|z|} e^{2\pi\Delta|\mathrm{Im}\, z|} \qquad \text{for all } z \in \mathbb{C}.$$

Proposition (Chandee and Vaaler)

Let $\Delta \geq 1$. Then $m_\Delta : \mathbb{C} \to \mathbb{C}$ is an even entire function such that:

$$(I) \quad \frac{-C}{1+x^2} \leq m_\Delta(x) \leq f(x), \text{ for some } C > 0 \text{ and for all } x \in \mathbb{R}.$$

$$(II) \quad m_\Delta(z) \ll \frac{\Delta^2}{1+\Delta|z|} e^{2\pi\Delta|\mathrm{Im}\, z|} \qquad \text{for all } z \in \mathbb{C}.$$

$$(III) \quad m_\Delta \in L^1(\mathbb{R}), \ \widehat{m_\Delta}(\xi) = 0 \text{ for } |\xi| \geq \Delta, \text{ and } \widehat{m_\Delta}(\xi) = O(1).$$

Proposition (Chandee and Vaaler)

Let $\Delta \geq 1$. *Then* $m_\Delta : \mathbb{C} \to \mathbb{C}$ *is an even entire function such that:*

$$(I) \quad \frac{-C}{1+x^2} \leq m_\Delta(x) \leq f(x), \text{ for some } C > 0 \text{ and for all } x \in \mathbb{R}.$$

$$(II) \quad m_\Delta(z) \ll \frac{\Delta^2}{1+\Delta|z|} e^{2\pi\Delta|\mathrm{Im}\, z|} \qquad \text{for all } z \in \mathbb{C}.$$

$$(III) \quad m_\Delta \in L^1(\mathbb{R}),\ \widehat{m_\Delta}(\xi) = 0 \text{ for } |\xi| \geq \Delta, \text{ and } \widehat{m_\Delta}(\xi) = O(1).$$

$$(IV) \int_{-\infty}^{\infty} \{f(x) - m_\Delta(x)\}\mathrm{d}x = \frac{1}{\Delta}\Big(2\log 2 - 2\log(1 + e^{-4\pi\Delta})\Big).$$

Proposition (Chandee and Vaaler)

Let $\Delta \geq 1$. Then $m_\Delta : \mathbb{C} \to \mathbb{C}$ is an even entire function such that:

$$(I) \quad \frac{-C}{1+x^2} \leq m_\Delta(x) \leq f(x), \text{ for some } C > 0 \text{ and for all } x \in \mathbb{R}.$$

$$(II) \quad m_\Delta(z) \ll \frac{\Delta^2}{1+\Delta|z|} e^{2\pi\Delta|\mathrm{Im}\, z|} \qquad \text{for all } z \in \mathbb{C}.$$

$$(III) \quad m_\Delta \in L^1(\mathbb{R}),\ \widehat{m_\Delta}(\xi) = 0 \text{ for } |\xi| \geq \Delta, \text{ and } \widehat{m_\Delta}(\xi) = O(1).$$

$$(IV) \int_{-\infty}^{\infty} \{f(x) - m_\Delta(x)\} \mathrm{d}x = \frac{1}{\Delta}\Big(2\log 2 - 2\log(1+e^{-4\pi\Delta})\Big).$$

$$(V) \ \left|m_\Delta(z)(1+|z|)^2\right| \ll 1 \quad \text{when } |\mathrm{Im}\, z| \leq \tfrac{1}{2} + \varepsilon \text{ and } |\mathrm{Re}\, z| \to \infty.$$

Then, for $t > 0$ sufficiently large

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1)$$
$$\leq \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1).$$

Then, for $t > 0$ sufficiently large

$$\begin{aligned}\log\left|\zeta\left(\frac{1}{2}+it\right)\right| &= \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1)\\ &\le \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1).\end{aligned}$$

Now, we apply the Guinand-Weil explicit formula for the function:

$$h(s) = m_{\Delta}(t-s).$$

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\left\{ \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right) - \log\pi \right\}\mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right)+\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right).$$

$$\sum_{\gamma} m_\Delta(t-\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(u)\left\{ \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right) - \log\pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n}+e^{-i\log n}\right)$$
$$+ m_\Delta\left(t-\frac{1}{2i}\right) + m_\Delta\left(t+\frac{1}{2i}\right).$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(u)\left\{\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right)-\log\pi\right\}\mathrm{d}u$$
$$=2\log t-\frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right)+O(1).$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(u)\left\{\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right)-\log\pi\right\}\mathrm{d}u$$
$$=2\log t-\frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right)+O(1).$$

$$m_\Delta\left(t-\frac{1}{2i}\right)+m_\Delta\left(t+\frac{1}{2i}\right)=O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).$$

We need to bound:

$$\frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_{\Delta}}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n}+e^{-i\log n}\right).$$

We need to bound:

$$\frac{1}{2\pi} \sum_{n \leq e^{2\pi\Delta}} \frac{\Lambda(n)}{\sqrt{n}} \; \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right) \left(e^{i \log n} + e^{-i \log n}\right).$$

We need to bound:

$$\frac{1}{2\pi} \sum_{n \leq e^{2\pi\Delta}} \frac{\Lambda(n)}{\sqrt{n}} \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n} + e^{-i\log n}\right).$$

The prime number theorem gives

$$\sum_{n \leq x} \Lambda(n) \sim x.$$

We need to bound:

$$\frac{1}{2\pi} \sum_{n \leq e^{2\pi\Delta}} \frac{\Lambda(n)}{\sqrt{n}} \; \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right) \left(e^{i \log n} + e^{-i \log n}\right).$$

The prime number theorem gives

$$\sum_{n \leq x} \Lambda(n) \sim x.$$

Then $\sum_{n \leq x} \Lambda(n) \ll x$, and this implies:

We need to bound:

$$\frac{1}{2\pi}\sum_{n\leq e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n}+e^{-i\log n}\right).$$

The prime number theorem gives

$$\sum_{n\leq x}\Lambda(n)\sim x.$$

Then $\sum_{n\leq x}\Lambda(n)\ll x$, and this implies:

$$\begin{aligned}\left|\frac{1}{2\pi}\sum_{n\leq e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n}+e^{-i\log n}\right)\right| &\ll \sum_{n\leq e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\\ &\ll e^{\pi\Delta}.\end{aligned}$$

Therefore:

$$\begin{aligned}\sum_{\gamma} m_\Delta(t-\gamma) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(u)\left\{\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right)-\log\pi\right\}\mathrm{d}u\\ &\quad -\frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n}+e^{-i\log n}\right)\\ &\quad + m_\Delta\left(t-\frac{1}{2i}\right)+m_\Delta\left(t+\frac{1}{2i}\right)\\ &= 2\log t-\frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right)+O(1)\\ &\quad + O\left(e^{\pi\Delta}\right)+O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).\end{aligned}$$

Then, for $t > 0$ sufficiently large

$$\begin{aligned}\log\left|\zeta\left(\frac{1}{2}+it\right)\right| &\le \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1)\\ &\le \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + O(1)\\ &\quad + O\left(e^{\pi\Delta}\right) + O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).\end{aligned}$$

Then, for $t > 0$ sufficiently large

$$\begin{aligned}\log\left|\zeta\left(\frac{1}{2}+it\right)\right| &\leq \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1)\\ &\leq \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + O(1)\\ &\quad + O\left(e^{\pi\Delta}\right) + O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).\end{aligned}$$

Finally, we choose $\pi\Delta = \log\log t - 3\log\log\log t$, and we obtain the desired result.