Class 18: Conditional bounds for $\zeta(s)$: part II

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Result of Littlewood

A classical result of Littlewood (1924) states that, under the Riemann hypothesis, there is $C > 0$ such that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll \exp\left(C\frac{\log t}{\log\log t}\right).$$

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for t sufficiently large. The order of magnitude has not been improved over the last ninety years, and the efforts have hence been concentrated in optimizing the values of the implicit constants.

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In this case $o(1) = \frac{\log\log\log t}{\log\log t}$.

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Lemma (Representation lemma)

Assume the Riemann hypothesis. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the function

$$f(x) = \log\left(\frac{4 + x^2}{x^2}\right).$$

Then, for $t > 0$ sufficiently large we have

$$\log\left|\zeta\left(\frac{1}{2} + it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t - \gamma) + O(1).$$

The sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ *be analytic in the strip* $|\mathrm{Im}\, s| \le \frac{1}{2} + \varepsilon$ *for some* $\varepsilon > 0$, *and assume that* $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ *for some* $\delta > 0$ *when* $|\mathrm{Re}\, s| \to \infty$. *Then*

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi} \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Connection to Fourier analysis

- We have written our object in consideration as

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

- From the explicit formula it would be very nice if we could find an special function m such that
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This is Beurling-Selberg's problem!!!

Developments of Beurling-Selberg's problem

Function	Optimal entire approximations
sgn(x)	Beurling 30's
$\chi_{[a,b]}(x)$	Selberg 50's and Logan 80's
$e^{-\lambda\|x\|}$	Graham-Vaaler '81
Even functions (e.g. $\log\|x\|$)	Carneiro-Vaaler '09
Even functions (e.g. $e^{-\lambda x^2}$) (Gaussian subordination)	Carneiro-Littmann-Vaaler '10
Odd functions (e.g. $sgn(x)e^{-\lambda x^2}$) (odd Gaussian subordination)	Carneiro-Vaaler '11

Theorem (Carneiro and Vaaler (TAMS))

Let ν *be a measure defined on the Borel sets of* $(0, \infty)$ *such that*

$$0 < \int_0^{\infty} \frac{\lambda}{\lambda^2 + 1} \, d\nu(\lambda) < \infty.$$

Define the function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *given by*

$$f(x) = \int_0^{\infty} \left\{ e^{-\lambda|x|} - e^{-\lambda} \right\} d\nu(\lambda),$$

where $f(0)$ *may take the value* ∞*.*

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where $f(0)$ may take the value ∞. Then, there exists a unique extremal minorant $G(z)$ of exponential type 2π for f. The function $G(x)$ interpolates the values of $f(x)$ at $\mathbb{Z}+\frac{1}{2}$.

$$G(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{f(n-\frac{1}{2})}{(z-n+\frac{1}{2})^2} + \sum_{n\in\mathbb{Z}} \frac{f'(n-\frac{1}{2})}{(z-n+\frac{1}{2})} \right\},$$

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$$\int_0^\infty \left\{ e^{-\lambda|x|} - e^{-\lambda} \right\} \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda$$
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$$\begin{aligned}\int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\mathrm{d}\lambda \\ &= \log\left(\frac{4\Delta^2+x^2}{x^2}\right) - \log(4\Delta^2+1) \\ &= f_\Delta(x),\end{aligned}$$

where

$$f_\Delta(x) = f\left(\frac{x}{\Delta}\right) - f\left(\frac{1}{\Delta}\right).$$

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Finally, we define

$$m_\Delta(x) = H_\Delta(\Delta x) \le f(x),$$

where $m_\Delta(z)$ is an entire function of exponential type $2\pi\Delta$.

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(*IV*) $\displaystyle\int_{-\infty}^{\infty} \{f(x) - m_\Delta(x)\}\mathrm{d}x = \frac{1}{\Delta}\Big(2\log 2 - 2\log(1 + e^{-4\pi\Delta})\Big)$.

(*V*) $\left|m_\Delta(z)(1+|z|)^2\right| \ll 1$ *when* $|\mathrm{Im}\, z| \leq \frac{1}{2} + \varepsilon$ *and* $|\mathrm{Re}\, z| \to \infty$.

Then, for $t > 0$ sufficiently large

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1)$$
$$\leq \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1).$$

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Now, we apply the Guinand-Weil explicit formula for the function:

$$h(s) = m_{\Delta}(t-s).$$

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\left\{ \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right) - \log\pi \right\}\mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right).$$

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$$m_\Delta\left(t-\frac{1}{2i}\right)+m_\Delta\left(t+\frac{1}{2i}\right)=O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).$$

We need to bound:

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$$\left|\frac{1}{2\pi}\sum_{n\le e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{i\log n}+e^{-i\log n}\right)\right|\ll\sum_{n\le e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}$$
$$\ll e^{\pi\Delta}.$$

Therefore:

$$\sum_{\gamma} m_\Delta(t-\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(u)\left\{ \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right) - \log\pi \right\} \mathrm{d}u$$
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$$+ m_\Delta\left(t-\frac{1}{2i}\right) + m_\Delta\left(t+\frac{1}{2i}\right)$$
$$= 2\log t - \frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + O\left(\frac{\Delta^2}{\sqrt{t}}\right) + O(1)$$
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$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| \le \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1)$$
$$\le \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + O\left(\frac{\Delta^2}{\sqrt{t}}\right) + O(1)$$
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Finally, we choose $\pi\Delta = \log\log t - 3\log\log\log t$, and we obtain the desired result.