$$\underline{h(s)} = m_\Delta(t-s)$$

$$m_\Delta(t-2i) \to X$$

$$\sum_\gamma h(\gamma) = \frac{1}{2\pi}\int_R h(u)\left\{ Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right) - \log\pi \right\} du$$

$$-\frac{1}{2\pi}\sum \frac{\Lambda(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{\log n}{2\pi}\right) + \hat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$

$$+h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

$$\sum_\gamma m_\Delta(t-\gamma) \;\Big|\; \boxed{\frac{1}{2\pi}\int_R m_\Delta(x)\left\{ Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{(t-x)}{2}\right) - \log\pi \right\} dx}$$

$$\hat{h}(\xi) = \int_R h(x)e^{-2\pi i\xi x}dx = \int_R m_\Delta(\underbrace{t-x}_{u})e^{-2\pi i\xi x}dx$$

$$= \int_R m_\Delta(u)e^{-2\pi i\xi(t-u)}du = e^{-2\pi i\xi t}\int_R m_\Delta(u)e^{2\pi i\xi u}du$$

$$\boxed{\hat{h}(\xi) = e^{-2\pi i\xi t}\,\widehat{m_\Delta}(-\xi)}$$

$$\to -\frac{1}{2\pi}\sum\frac{\Lambda(n)}{\sqrt{n}}\left\{\widehat{m_\Delta}\left(\frac{-\log n}{2\pi}\right)e^{-2\pi i\left(\frac{\log n}{2\pi}\right)t} + \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)e^{-2\pi i\left(\frac{\log n}{2\pi}\right)t}\right.$$

$$\Rightarrow -\frac{1}{2\pi}\sum\frac{\Lambda(n)}{\sqrt{n}}\widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left\{e^{-i(\log n)t} + e^{i(\log n)t}\right\} \to \Delta$$

$$\to \boxed{m_\Delta\left(t-\frac{1}{2i}\right) + m_\Delta\left(t+\frac{1}{2i}\right)}$$

$$\left|m_\Delta\left(t-\frac{1}{2i}\right)\right| \le C.\frac{\Delta^2}{1+\Delta\left|t-\frac{1}{2i}\right|}e^{2\pi\Delta\left|\frac{1}{2}\right|}$$

$$= C.\frac{\Delta^2 e^{\pi\Delta}}{1+\Delta t}$$

$$m_\Delta\left(t-\frac{1}{2i}\right) = O\left(\frac{\Delta^2 e^{\pi\Delta}}{1+\Delta t}\right) \to \Delta$$

$$m_\Delta\left(t+\frac{1}{2i}\right) = O\left(\frac{\Delta^2 e^{\pi\Delta}}{1+\Delta t}\right)$$

$$\frac{1}{2\pi}\int_{\mathbb{R}} m_\Delta(u)\left\{\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{(t-u)}{2}\right)-\log\pi\right\}du$$

(III)

$$=\frac{1}{2\pi}\int_{R} m_\Delta(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{(t-u)}{2}\right)+O(1)$$

$$\frac{\Gamma'}{\Gamma}(s)=\log s+O\left(\frac{1}{|s|}\right)$$

$$\mathrm{Re}\frac{\Gamma'}{\Gamma}(s)=\log|s|+O\left(\frac{1}{|s|}\right)$$

$$\boxed{\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+ix\right)=\log\left|\frac{1}{4}+ix\right|+O\left(\frac{1}{|\frac{1}{4}+ix|}\right)}$$

$$\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+ix\right)=\log\left|\frac{1}{4}+ix\right|+O(1)$$

$$\boxed{|x|\leq 2\Rightarrow \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+ix\right)=O(1)}$$

$$|x|\geq 2\Rightarrow \log\left|\frac{1}{4}+ix\right|=\frac{1}{2}\log\left(\frac{1}{16}+x^2\right)$$

$$=\frac{1}{2}\log\left(x^2\left(1+\frac{1}{16x^2}\right)\right)=\log|x|+O\left(\frac{1}{x^2}\right)$$

$$|x|\geq 2\quad \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+ix\right)=\log|x|+O(1)$$

$$\boxed{\left|\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+ix\right)\right|\leq C\log(|x|+2)\quad \forall x\in\mathbb{R}}$$

$$\underbrace{\frac{1}{2\pi}\int_{4\sqrt{t}}^{\infty} m_\Delta(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}}+\frac{1}{2\pi}\int_{-\infty}^{-4\sqrt{t}} m_\Delta(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}+\frac{1}{2\pi}\int_{-4\sqrt{t}}^{4\sqrt{t}} m_\Delta\,\mathrm{Re}\frac{\Gamma'}{\Gamma}$$

$$\left|\int_{4\sqrt{t}}^{\infty} m_\Delta(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}\cdots\right|\leq\int_{4\sqrt{t}}^{\infty}|m_\Delta(u)|\cdot C\log(|t-u|+2)\,du$$

$$-\frac{C}{x^2}\leq m_\Delta\leq f(x)=\log\left(\frac{4+x^2}{x^2}\right)=\log\left(1+\frac{4}{x^2}\right)\leq\frac{4}{x^2}$$

$$\cancel{|m_\Delta|\leq\frac{C}{x^2}}\qquad |m_\Delta(x)|\leq\frac{C}{x^2}$$

$$\int_{4\sqrt{t}}^{\infty}\frac{1}{u^2}\log(|t-u|+2)\,du$$

$$\int_{4\sqrt{t}}^{\infty} \frac{1}{u^2} \log(|t-u|+2)\,du$$

$$u \geq 4\sqrt{t} \Rightarrow u^2 \geq 16t$$

$$|t-u| \leq |t| + |u| \leq t + u$$
$$\leq \frac{u^2}{16} + u \leq u^3$$
$$|t-u|+2 \leq u^3 + 2 \leq u^4$$
$$\log(|t-u|+2) \leq 4\log u$$

$$\int_{4\sqrt{t}}^{\infty} \frac{4\log u\,du}{u^2} = 4\int_{4\sqrt{t}}^{\infty} \frac{\log u}{u^2}\,du$$
$$\leq 4\int_{4}^{\infty} \frac{\log u}{u^2}\,du \leq C$$

$$\left|\underbrace{\int_{4\sqrt{t}}^{\infty} m_\Delta(u)\, \mathrm{Re}\,\frac{\Gamma'}{\Gamma}}\right| \leq C$$
$$\Rightarrow \int_{4\sqrt{t}}^{\infty} = O(1)$$

$$\left|\int_{-\infty}^{-4\sqrt{t}} m_\Delta\, \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\right| \leq C \Rightarrow \int_{-\infty}^{-4\sqrt{t}} m_\Delta\, \mathrm{Re}\,\frac{\Gamma'}{\Gamma} = O(1)$$

$$\int_{-4\sqrt{t}}^{4\sqrt{t}} m_\Delta(x)\, \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\left(\frac{t-x}{2}\right)\right)dx$$
$$= \int_{-4\sqrt{t}}^{4\sqrt{t}} m_\Delta(x)\left\{\underbrace{\log\left|\frac{1}{4}+i\left(\frac{t-x}{2}\right)\right|}_{|x|\leq 4\sqrt{t}} + O\left(\frac{1}{|t-x|}\right)\right\}dx$$

$$\log\left|\frac{1}{4}+i\left(\frac{t-x}{2}\right)\right| = \frac{1}{2}\ln\left(\left(\frac{t-x}{2}\right)^2 + \frac{1}{16}\right)$$
$$= \frac{1}{2}\ln\left(\left(\frac{t-x}{2}\right)^2\left\{1+\frac{1}{16\left(\frac{t-x}{2}\right)^2}\right\}\right)$$
$$= \frac{1}{2}\ln\left(\left(\frac{t-x}{2}\right)^2\right) + \frac{1}{2}\ln\left(1+\frac{1}{4|t-x|^2}\right)$$
$$= \ln\left(\frac{t-x}{2}\right) + O\left(\frac{1}{|t-x|^2}\right)$$
$$= \ln\left(\frac{t}{2}\right) + \ln\left(\frac{t-x}{t}\right) + O(\ )$$
$$= \ln\left(\frac{t}{2}\right) + \ln\left(1-\frac{x}{t}\right) + O(\ )$$

$|x| \le 4\sqrt{t}$ ③

$$\log\left|\frac{1}{4}+i\left(\frac{t-x}{2}\right)\right| = \ln\left(\frac{t}{2}\right)+\ln\left(1-\frac{x}{t}\right)+O\left(\frac{1}{|t-x|^2}\right)$$

$$= \ln\left(\frac{t}{2}\right)+O\left(\frac{|x|}{t}\right)+O\left(\frac{1}{|t-x|^2}\right)$$

$$= \ln\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)+O\left(\frac{1}{t^2}\right)$$

$$|t-x| \ge t-|x| \ge t-4\sqrt{t} \ge \frac{t}{2}$$

$(t\to\infty)$

$$\Rightarrow \log\left|\frac{1}{4}+i\left(\frac{t-x}{2}\right)\right| = \log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)$$

$$\int_{-4\sqrt{t}}^{4\sqrt{t}} m_\Delta(x)\left\{\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)+\left(\frac{1}{t}\right)\right\}dx$$

$$= \int_{-4\sqrt{t}}^{4\sqrt{t}} m_\Delta(x)\underbrace{\left\{\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)\right\}}dx$$

$$= \left(\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)\right)\int_{-4\sqrt{t}}^{4\sqrt{t}} m_\Delta(x)\,dx$$

$$= \left(\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)\right)\cdot\left(\int_{-\infty}^{\infty} m_\Delta(x)\,dx-\int_{4\sqrt{t}}^{\infty} m_\Delta-\int_{-\infty}^{-4\sqrt{t}} m_\Delta\right)$$

$$= \left(\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)\right)\int_{-\infty}^{\infty} m_\Delta(x)\,dx$$

$$+O\left(\left(\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)\right)\frac{1}{\sqrt{t}}\right)$$

$$\int_{4\sqrt{t}}^{\infty} m_\Delta(x)\,dx \le \int_{4\sqrt{t}}^{\infty}\frac{c}{x^2}\,dx \le \frac{c}{\sqrt{t}}$$

$$= \left(\log\left(\frac{t}{2}\right)+O\left(\frac{1}{\sqrt{t}}\right)\right)\int_{-\infty}^{\infty} m_\Delta(x)\,dx+O\left(\frac{\log t}{\sqrt{t}}\right)$$

$$= \left(\log t-\log 2+O\left(\frac{1}{\sqrt{t}}\right)\right)\widehat{m_\Delta}(0)+O(\ )$$

$$= \log t\cdot\widehat{m_\Delta}(0)+O(1)$$

$$= \cancel{\log t\cdot\int_{-\infty}^{\infty} m_\Delta(x) - f(\ )}$$

$$= \log t\cdot\left\{\int_{-\infty}^{\infty}(m_\Delta(x)-f(x))\,dx+\int_{-\infty}^{\infty} f(x)\,dx\right\}+O(1)$$

$$= \log t\left(\int^{\infty} m_\Delta(x)-f(x)\right)dx+\log t\cdot 4\pi+O(1)$$

$$\frac{1}{2\pi}\int_{R} m_\Delta(u)\left\{\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}-\log\pi\right)\right\}du$$

$$=\frac{1}{2\pi}\left[\log t\cdot\left\{\int_{-\infty}^{\infty} m_\Delta - f\right\}+\log t\cdot 4\pi+O(1)\right]$$

$$=2\log t+\frac{\log t}{2\pi}\cdot\left[\frac{2}{\Delta}\log 2-\frac{2}{\Delta}\log\left(1+e^{-4\pi\Delta}\right)\right]$$

$$+O(1)$$

$$=2\log t+\frac{\log t}{\pi\Delta}\left(\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right)\right)+O(1)$$

$$\left|\frac{\log n}{2\pi}\right|\leq\Delta$$

$$\log n\leq 2\pi\Delta$$

$$\boxed{n\leq e^{2\pi\Delta}}$$

INTEGRATION BY PARTS

$$\sum_{n\leq x}\frac{\Lambda(n)}{\sqrt{n}}=\int_{2^-}^{x^+}\frac{d\left(\sum_{n\leq y}\Lambda(n)\right)}{\sqrt{y}}$$

$$=\frac{\sum_{n\leq x}\Lambda(n)}{\sqrt{x}}+\frac{1}{2}\int_2^x\left(\sum_{n\leq y}\Lambda(n)\right)\frac{dy}{y^{3/2}}$$

$$\leq\frac{c.x}{\sqrt{x}}+\frac{c}{2}\int_2^x\frac{y}{y^{3/2}}\,dy$$

$$=c\sqrt{x}+\frac{c}{2}\cdot(\sqrt{x}-\sqrt{2})$$

$$\ll\sqrt{x}$$

$$\sum_{n\leq e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\ll e^{\pi\Delta}$$

We want to bound (well written)

$$\sum_{n \leq x} \frac{\Lambda(n)}{\sqrt{n}}$$

Note that if $F(x) = \sum_{n \leq x} \Lambda(n)$,

the P.N.T gives $F(x) \leq c.x \quad \forall x \geq 2$

$$\Rightarrow \sum_{n \leq x} \frac{\Lambda(n)}{\sqrt{n}} = \int_{2}^{x^+} \frac{dF(y)}{\sqrt{y}}$$

$$= \frac{F(x)}{\sqrt{x}} - \frac{F(2^-)}{\sqrt{2^-}} - \int_2^x F(y) \cdot \left\{ \frac{1}{\sqrt{y}} \right\}' dy$$

$$= \frac{F(x)}{\sqrt{x}} + \frac{1}{2} \int_2^x \frac{F(y)}{y^{3/2}} dy$$

$$\leq \frac{c.x}{\sqrt{x}} + \frac{1}{2} \int_2^x \frac{c\,y}{y^{3/2}} dy$$

$$= c\sqrt{x} + \frac{c}{2} \int_2^x y^{-1/2} dy$$

$$= c\sqrt{x} + \frac{c}{2} \left[\frac{x^{1/2}}{1/2} - \frac{2^{1/2}}{1/2} \right]$$

$$= c\sqrt{x} + c\sqrt{x} - c\sqrt{2} \leq 2c\sqrt{x}$$

$$\Rightarrow \sum_{n \leq x} \frac{\Lambda(n)}{\sqrt{n}} \leq 2c\sqrt{x}$$