Class 19: Advances on zeta and gaps between zeros of zeta

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08-November-2021

Result of Littlewood

A classical result of Littlewood (1924) states that, under the Riemann hypothesis, there is $C > 0$ such that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll \exp\left(C\frac{\log t}{\log\log t}\right).$$

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for t sufficiently large. The order of magnitude has not been improved over the last ninety years, and the efforts have hence been concentrated in optimizing the values of the implicit constants.

Assuming the Riemann hypothesis, we have

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(2) Soundararajan (2009) : $C = 0.373$.
(3) Chandee and Soundararajan (2011) : $C = \frac{\ln(2)}{2} \approx 0.347$.
In this case $o(1) = \frac{\log\log\log t}{\log\log t}$.

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Lemma (Representation lemma)

Assume the Riemann hypothesis. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *be the function*

$$f(x) = \log\left(\frac{4 + x^2}{x^2}\right).$$

Then, for $t > 0$ *sufficiently large we have*

$$\log\left|\zeta\left(\frac{1}{2} + it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t - \gamma) + O(1).$$

The sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ *of* $\zeta(s)$.

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ *be analytic in the strip* $|\mathrm{Im}\, s| \leq \frac{1}{2} + \varepsilon$ *for some* $\varepsilon > 0$, *and assume that* $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ *for some* $\delta > 0$ *when* $|\mathrm{Re}\, s| \to \infty$. *Then*

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Connection to Fourier analysis

- We have written our object in consideration as

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1).$$

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This is Beurling-Selberg's problem!!!

Theorem (Carneiro and Vaaler (TAMS))

Let ν be a measure defined on the Borel sets of $(0, \infty)$ such that

$$0 < \int_0^\infty \frac{\lambda}{\lambda^2 + 1} \, d\nu(\lambda) < \infty.$$

Define the function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ given by

$$f(x) = \int_0^\infty \left\{ e^{-\lambda|x|} - e^{-\lambda} \right\} d\nu(\lambda),$$

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where $f(0)$ may take the value ∞. Then, there exists a unique extremal minorant $G(z)$ of exponential type 2π for f. The function $G(x)$ interpolates the values of $f(x)$ at $\mathbb{Z} + \frac{1}{2}$.

$$G(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{f(n - \frac{1}{2})}{(z - n + \frac{1}{2})^2} + \sum_{n\in\mathbb{Z}} \frac{f'(n - \frac{1}{2})}{(z - n + \frac{1}{2})} \right\},$$

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(IV) $\displaystyle\int_{-\infty}^{\infty} \{f(x) - m_\Delta(x)\} \mathrm{d}x = \frac{1}{\Delta}\left(2\log 2 - 2\log(1+e^{-4\pi\Delta})\right)$.

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$$(IV) \int_{-\infty}^{\infty} \{f(x) - m_\Delta(x)\} \mathrm{d}x = \frac{1}{\Delta}\Big(2\log 2 - 2\log(1 + e^{-4\pi\Delta})\Big).$$

$$(V) \quad \left|m_\Delta(z)(1+|z|)^2\right| \ll 1 \quad \text{when } |\mathrm{Im}\, z| \leq \tfrac{1}{2} + \varepsilon \text{ and } |\mathrm{Re}\, z| \to \infty.$$

Then, for $t > 0$ sufficiently large

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1)$$
$$\leq \log t - \frac{1}{2}\sum_{\gamma} m_{\Delta}(t-\gamma) + O(1).$$

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Now, we apply the Guinand-Weil explicit formula for the function:

$$h(s) = m_{\Delta}(t-s).$$

$$\sum_{\gamma} m_{\Delta}(t-\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}(u)\left\{ \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right) - \log\pi\right\}\mathrm{d}u$$

$$-\frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_{\Delta}}\left(\frac{\log n}{2\pi}\right)\left(e^{it\log n}+e^{-it\log n}\right)$$

$$+\, m_{\Delta}\left(t-\frac{1}{2i}\right) + m_{\Delta}\left(t+\frac{1}{2i}\right).$$

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$$m_\Delta\left(t-\frac{1}{2i}\right)+m_\Delta\left(t+\frac{1}{2i}\right)=O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).$$

We need to bound:

$$\frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\;\widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{it\log n}+e^{-it\log n}\right).$$

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$$\left|\frac{1}{2\pi}\sum_{n\le e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\ \widehat{m_\Delta}\left(\frac{\log n}{2\pi}\right)\left(e^{it\log n}+e^{-it\log n}\right)\right|\ll\sum_{n\le e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}$$

$$\ll e^{\pi\Delta}.$$

Therefore:

$$\sum_{\gamma} m_\Delta(t-\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(u)\left\{ \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i(t-u)}{2}\right) - \log\pi \right\} \mathrm{d}u$$

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$$+ m_\Delta\left(t-\frac{1}{2i}\right) + m_\Delta\left(t+\frac{1}{2i}\right)$$

$$= 2\log t - \frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + O(1)$$

$$+ O\Big(e^{\pi\Delta}\Big) + O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right).$$

Then, for $t > 0$ sufficiently large

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Finally, we choose $\pi\Delta = \log\log t - 3\log\log\log t$, and we obtain the desired result.

The critical line

1. **Mean value**: Watt 2010

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2 \mathrm{d}t = T\log T - (1+\log 2\pi - 2\gamma)T + O(T^{131/416+\varepsilon}).$$

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3. **Omega results**: R. de la Bretèche and Tenenbaum 2018

$$\max_{t\in[1,T]} |\zeta(\tfrac{1}{2}+it)| \geq \exp\left((\sqrt{2}+o(1))\frac{(\log T)^{1/2}(\log\log\log T)^{1/2}}{(\log\log T)^{1/2}}\right).$$

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4. **Conditional bounds**: Chandee and Soundararajan 2011
$$|\zeta(\tfrac{1}{2}+it)| \le \exp\left((\frac{\log 2}{2}+o(1))\frac{\log t}{\log\log t}\right).$$

Conjecture for $\zeta(\frac{1}{2}+it)$

Farmer, Gonek, Hughes (2007): As $T \to \infty$

$$\max_{t\in[1,T]} |\zeta(\tfrac{1}{2}+it)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)(\log T)^{1/2}(\log\log T)^{1/2}\right).$$

The edge of the critical strip

1. **Mean value**: Balasubramanian, K. Ramachandra 1992

$$\int_1^T |\zeta(1+it)|^2 \mathrm{d}t = \zeta(2)\,T - \pi \log T + O\big((\log T)^{2/3}(\log\log T)^{1/3}\big).$$

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4. **Conditional bounds**: Lamzouri, Li and Soundararajan 2016

$$|\zeta(1+it)| \le 2e^{\gamma} \left(\log\log t - \log 2 + \frac{1}{2} + \frac{1}{\log\log t} \right).$$

Conjecture for $\zeta(1+it)$

Granville and Soundararajan (2006): As $T \to \infty$

$$\max_{t\in[T,2T]} |\zeta(1+it)| = e^{\gamma}\Big(\log\log T + \log\log\log T + C_1 + o(1)\Big).$$

The argument function $S(t)$

Mean value:

1. Unconditionally, Selberg 1946:

$$\int_1^T |S(t)|^2 \mathrm{d}t = \frac{T}{2\pi^2} \log\log T + O\big(T(\log\log T)^{1/2}\big).$$

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2. Conditionally, Goldston 1987 (Selberg):

$$\int_1^T |S(t)|^2 \mathrm{d}t = \frac{T}{2\pi^2} \log\log T + O(T).$$

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Unconditional bounds: Trudgian (Backlund) 2002

$$|S(t)| \leq (0.112 + o(1)) \log t.$$

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Unconditional omega results: Tsang 1986

$$S(t) = \Omega_{\pm}\left(\frac{(\log t)^{1/3}}{(\log \log t)^{1/3}}\right).$$

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2. Fujii (2004) : $C_0 = 0.67$.

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1. Ramachandra and Sankaranarayanan (1993) : $C_0 = 1.119$.
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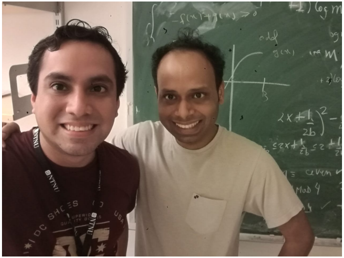

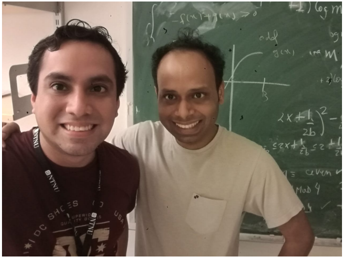

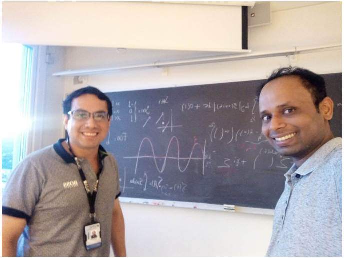

Conjecture for $S(t)$

Farmer, Gonek, Hughes (2007): As $T \to \infty$

$$\limsup_{t\to\infty} \frac{S(t)}{(\log t)^{1/2}(\log\log t)^{1/2}} = \frac{1}{\pi\sqrt{2}}.$$

The function $S_1(t)$

For $t > 0$ define

$$S_1(t) = \int_0^t S(\tau)\,\mathrm{d}\tau + \delta_1,$$

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2. Conditionally, Chirre and Quesada-Herrera 2021:

$$\int_1^T |S_1(t)|^2 \mathrm{d}t = \frac{C_1}{2\pi^2} T + O\left(\frac{T}{(\log T)^2}\right).$$


Andrés Chirre

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Unconditional omega results: Tsang 1986, 1993

$$S_1(t) = \Omega_+\left(\frac{(\log t)^{1/2}}{(\log\log t)^{3/2}}\right),$$

and

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Ω_- result for $S_1(t)$ is an open problem

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5. $\dfrac{\zeta'}{\zeta}(\sigma + it)$.

Gaps between zeros of zeta

Let $0 < \gamma_1 \leq \gamma_2 \leq \ldots$ be the consecutive ordinates of the non-trivial zeros of $\zeta(s)$ with positive imaginary parts (counting multiplicity).

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From Problem (35) we have unconditionally that,

$$\gamma_{n+1} - \gamma_n = O(1).$$

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we can improve the previous result related to gaps (conditionally). For any $\varepsilon > 0$, there is n_0 such that if $n \geq n_0$:

$$\gamma_{n+1} - \gamma_n \leq \frac{\pi + \varepsilon}{\log \log \gamma_n}.$$

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3. Unconditional: for some $A > 0$ we have

$$\gamma_{n+1} - \gamma_n \leq \frac{A}{\log \log \log \gamma_n}.$$