$$\leq \frac{\log t}{2\pi\Delta}\cdot \log 2 - \frac{\log t}{2\pi\Delta}\log\left(1+e^{-4\pi\Delta}\right)$$

$$+O(1)+O\left(e^{\pi\Delta}\right)+O\left(\frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}\right)$$

$$\boxed{\frac{\log\log t}{2}\leq \pi\Delta = \log\log t - 3\log\log\log t < \log\log t}$$

$$\leq \frac{\log 2}{2}\cdot\frac{\log t}{\log\log t - 3\log\log\log t}$$

$$+O\left(\frac{\log t}{\log\log t}\cdot\frac{1}{e^{4\pi\Delta}}\right)$$

$$+O(1)+O\left(e^{\log\log t - 3\log\log\log t}\right)$$

$$+O\left(\frac{(\log\log t)^2}{t}e^{\pi\Delta}\right) \qquad e^{3\log(\log t)^3}$$

$$=\frac{\log 2}{2}\left\{\frac{\log t}{\log\log t}+\frac{\log t}{\log\log t - 3\log\log\log t}-\frac{\log t}{\log\log t}\right\}$$

$$+O\left(\frac{\log t}{\log\log t}\cdot\frac{1}{e^{2\log(\log t)^2}}\right)+O(1)$$

$$+O\left(\frac{\log t}{(\log\log t)^3}\right)+O\left(\frac{(\log\log t)^2}{t}\frac{\log t}{(\log\log t)^3}\right)$$

$$=\frac{\log 2}{2}\left(\frac{\log t}{\log\log t}\right)+O\left(\frac{\log t\cdot\log\log\log t}{(\log\log t - 3\log\log\log t)(\log\log t)}\right)$$

$$+O\left(\frac{1}{\log t\cdot\log\log t}\right)+O(1)+O\left(\frac{\log t}{(\log\log t)^3}\right)+O\left(\frac{\log t}{t\log\log t}\right)$$

$$= \cancel{\frac{\log 2}{2}+O\left(\frac{\log\log t}{\ldots}\right)\ \frac{\log t}{\ldots}}$$

$$= \frac{\log 2}{2} \cdot \left(\frac{\log t}{\log\log t}\right) + O\left(\frac{\log t \cdot \log\log\log t}{(\log\log t)^2}\right)$$

$$+ O(1) + O\left(\frac{\log t}{(\log\log t)^3}\right) + O\left(\frac{\log t}{t \log\log t}\right)$$

$$= \left(\frac{\log 2}{2} + O\left(\frac{\log\log\log t}{\log\log t}\right)\right)\frac{\log t}{\log\log t}$$

$$\Rightarrow \log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \left(\frac{\log 2}{2} + o(1)\right)\frac{\log t}{\log\log t}$$

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \exp\left(\left(\frac{\log 2}{2} + o(1)\right)\frac{\log t}{\log\log t}\right)$$

$$S(t) = \Omega_+\left(\frac{(\log t)^{1/3}}{(\log\log t)^{1/3}}\right)$$

$$S(t_n) \geq C \cdot \frac{(\log t_n)^{1/3}}{(\log\log t_n)^{1/3}} \qquad t_n \to \infty$$

$$S(t) = \Omega_-\left(\frac{(\log t)^{1/3}}{(\log\log t)^{1/3}}\right)$$

$$S(t_n) \leq -C \cdot \frac{(\log t_n)^{1/3}}{(\log\log t_n)^{1/3}} \;;\; t_n \to \infty$$

$$S(t) = \Omega\left(\frac{(\log t)^{1/2}(\log\log\log t)^{1/2}}{(\log\log t)^{1/2}}\right)$$

$$|S(t_n)| \geq C \frac{(\log t_n)^{1/2}(\log\log\log t_n)^{1/2}}{(\log\log t_n)}$$

$$N(t) = \frac{t}{2\pi}\ln\left(\frac{t}{2\pi}\right) - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right)$$

$$N(t) = \int_0^t \frac{1}{2\pi}\ln\left(\frac{x}{2\pi}\right)dx + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right)$$

$$N(t+h) - N(t)$$

$$= \int_t^{t+h} \frac{1}{2\pi}\ln\left(\frac{x}{2\pi}\right)dx + S(t+h) - S(t) + O\left(\frac{1}{t}\right)$$

$$\geq \frac{h}{2\pi}\ln\left(\frac{t}{2\pi}\right) + \underline{S(t+h) - S(t) + O\left(\frac{1}{t}\right)}$$

$|h| \leq 1$. Let $\varepsilon > 0$:

$$|S(t+h) - S(t)| \leq |S(t+h)| + |S(t)|$$

$$\leq \left(\frac{1}{4}+\varepsilon\right)\frac{\log(t+h)}{\log\log(t+h)} \leq \left(\frac{1}{4}+\varepsilon\right)\frac{\log t}{\log\log t}$$

$$\leq \left(\frac{1}{4}+\varepsilon\right)\frac{\log(t+h)}{\log\log t} + \left(\frac{1}{4}+\varepsilon\right)\frac{\log t}{\log\log t}$$

$\boxed{\log(t+h) \leq \log t + O\left(\frac{h}{t}\right)}$

$$= \left(\frac{1}{2}+2\varepsilon\right)\frac{\log t}{\log\log t} + \left(\frac{1}{4}+\varepsilon\right)\frac{1}{t\log\log t}$$

$\boxed{\log(t+h) \leq \log t + \frac{h}{t}}$

$$= \left(\frac{1}{2}+2\varepsilon\right)\frac{\log t}{\log\log t} + \underbrace{\left(\frac{1}{4}+\varepsilon\right)\frac{1}{t\log t}}_{<\varepsilon}\left(\frac{\log t}{\log\log t}\right)$$

$$\leq \left(\frac{1}{2}+3\varepsilon\right)\frac{\log t}{\log\log t}$$

$$\left|S(t+h) - S(t) + O\left(\frac{1}{t}\right)\right| \leq \left(\frac{1}{2}+3\varepsilon\right)\frac{\log t}{\log\log t} + \frac{C}{t}$$

$$= \left(\frac{1}{2}+3\varepsilon\right)\left(\frac{\log t}{\log\log t}\right) + \left(\frac{\log t}{\log\log t}\right)\underbrace{\left(\frac{\log\log t \cdot C}{t \log t}\right)}_{\to 0}$$

$$\leq \left(\frac{1}{2}+4\varepsilon\right)\frac{\log t}{\log\log t}$$

$N(t+h) - N(t)$

$$\geq \frac{h}{2\pi} \ln\left(\frac{t}{2\pi}\right) + \underbrace{S(t+h) - S(t) + O\left(\frac{1}{t}\right)}$$

$$\left| S(t+h) - S(t) + O\left(\frac{1}{t}\right) \right| \leq \left(\frac{1}{2} + 4\varepsilon\right) \frac{\log t}{\log\log t}$$

$$\geq \frac{h}{2\pi} \ln t - \frac{h \ln(2\pi)}{2\pi} - \left(\frac{1}{2} + 4\varepsilon\right) \frac{\log t}{\log\log t}$$

$$\geq \underbrace{\frac{h}{2\pi} \ln t - \left(\frac{1}{2} + 5\varepsilon\right) \frac{\log t}{\log\log t}}_{>0}$$

$$\frac{h}{2\pi} \ln t > \left(\frac{1}{2} + 5\varepsilon\right) \frac{\log t}{\log\log t}$$

$$\boxed{h > \frac{(\pi + 10\pi\varepsilon)}{\log\log t}}$$

$$\boxed{h = \frac{\pi + 11\pi\varepsilon}{\log\log t}}$$

$$N(t+h) - N(t) > 0$$

$t = \gamma_n$

$$N(h + \gamma_n) - N(\gamma_n) > 0$$

$$\gamma_n < \gamma \leq h + \gamma_n$$

$$\Rightarrow \gamma_n < \gamma_{n+1} \leq \gamma_n + \frac{\pi + 11\pi\varepsilon}{\log\log \gamma_n}$$

$$\gamma_{n+1} - \gamma_n \leq \frac{\pi + \varepsilon}{\log\log \gamma_n}$$