Class 20: Unconditional gaps between zeros of zeta

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11-November-2021

The critical line

1. **Mean value**: Watt 2010

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2 \mathrm{d}t = T\log T - (1+\log 2\pi - 2\gamma)T + O(T^{131/416+\varepsilon}).$$

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4. **Conditional bounds**: Chandee and Soundararajan 2011

$$|\zeta(\tfrac{1}{2}+it)| \leq \exp\left((\frac{\log 2}{2}+o(1))\frac{\log t}{\log\log t}\right).$$

Conjecture for $\zeta(\frac{1}{2}+it)$

Farmer, Gonek, Hughes (2007): As $T \to \infty$

$$\max_{t\in[1,T]} |\zeta(\tfrac{1}{2}+it)| = \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)(\log T)^{1/2}(\log\log T)^{1/2}\right).$$

The argument function $S(t)$

Mean value:

1. Unconditionally, Selberg 1946:

$$\int_1^T |S(t)|^2 \mathrm{d}t = \frac{T}{2\pi^2} \log\log T + O\big(T(\log\log T)^{1/2}\big).$$

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2. Conditionally, Goldston 1987 (Selberg):

$$\int_1^T |S(t)|^2 \mathrm{d}t = \frac{T}{2\pi^2} \log\log T + O(T).$$

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Unconditional bounds: Trudgian (Backlund) 2002

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Unconditional omega results: Tsang 1986

$$S(t) = \Omega_{\pm}\left(\frac{(\log t)^{1/3}}{(\log\log t)^{1/3}}\right).$$

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2. Fujii (2004) : $C_0 = 0.67$.
3. Goldston and Gonek (2007) : $C_0 = 0.5$.
4. Carneiro, Chandee and Milinovich (2012) : $C_0 = 0.25$.

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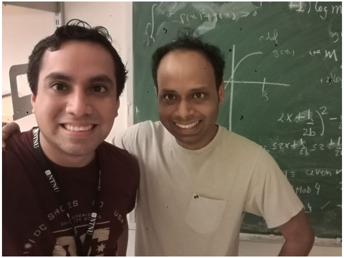
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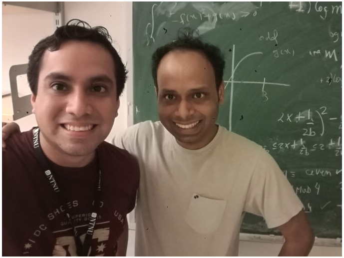

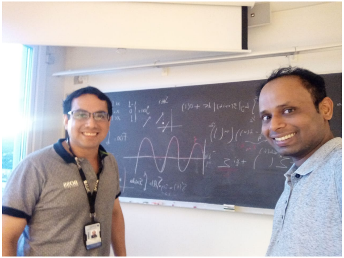

Conjecture for $S(t)$

Farmer, Gonek, Hughes (2007): As $T \to \infty$

$$\limsup_{t\to\infty} \frac{S(t)}{(\log t)^{1/2}(\log\log t)^{1/2}} = \frac{1}{\pi\sqrt{2}}.$$

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4. $\dfrac{1}{\zeta(1 + it)}$
5. $\dfrac{\zeta'}{\zeta}(\sigma + it)$.

Gaps between zeros of zeta

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From Problem (35) we have unconditionally that,

$$\gamma_{n+1} - \gamma_n = O(1).$$

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$$|S(t)| \leq \left(\frac{1}{4} + o(1)\right) \frac{\log t}{\log\log t},$$

we can improve the previous result related to gaps (conditionally). For any $\varepsilon > 0$, there is n_0 such that if $n \geq n_0$:

$$\gamma_{n+1} - \gamma_n \leq \frac{\pi + \varepsilon}{\log\log \gamma_n}.$$

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2. Conditional:

$$\gamma_{n+1} - \gamma_n \leq \frac{\pi + \varepsilon}{\log\log\gamma_n}.$$

3. Unconditional: for some $A > 0$ we have

$$\gamma_{n+1} - \gamma_n \leq \frac{A}{\log\log\log\gamma_n}.$$

Theorem (Borel-Carathéodory theorem)

Let $\Omega \subset \mathbb{C}$ be an open set such that contains the disc $|z| \leq R$. Then, for $0 < r < R$ we have that

$$\max_{|z|\leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z|=R} \operatorname{Re} f(z) + \frac{R+r}{R-r} |f(0)|.$$

Theorem (Hadamard's three-circles theorem)

Let $\Omega \subset \mathbb{C}$ be an open set such that contains the annulus $r_1 \leq |z| \leq r_3$. Let $f : \Omega \to \mathbb{C}$ be an holomorphic function. Define $M(r)$ the maximum of $|f(z)|$ on the circle $|z| = r$. Then, for $r_1 < r_2 < r_3$ we have:

$$(M_2)^{\log(r_3/r_1)} \leq (M_1)^{\log(r_3/r_2)}(M_3)^{\log(r_2/r_1)}.$$

Theorem

Let T sufficiently large. Then, there is a zero $\rho = \beta + i\gamma$ of $\zeta(s)$ such that

$$|\gamma - T| \leq \frac{A}{\log \log \log T},$$

for some universal constant $A > 0$.

Suppose that $\zeta(s)$ has no zeros in $T - \delta \le \mathrm{Im}\, s \le T + \delta$, with $0 < \delta < \frac{1}{2}$. Define the function $\log \zeta(s)$, analytic in $-2 \le \mathrm{Re}\, s \le 3$ and $T - \delta \le \mathrm{Im}\, s \le T + \delta$.

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Let c_ν, C_ν, $\mathcal{C}_\nu$ and Γ_ν be four concentric circles, with centre $2-\frac{\nu}{4}\delta+iT$ and radii $\frac{\delta}{4}$, $\frac{\delta}{2}$, $\frac{3\delta}{4}$ and δ respectively. Consider these set of circles such for $\nu=0,1,...,n$ where $n=[12/\delta]+1$, so that $2-\frac{n}{4}\delta \leq -1$.

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Define m_ν, M_ν, $\mathcal{M}_\nu$ and Γ_ν the maxima of $|\log\zeta(s)|$ on the circles c_ν, C_ν, and $\mathcal{C}_\nu$ respectively.

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We have for all circles that

$$\mathrm{Re}\,\{\log\zeta(s)\} = \log|\zeta(\sigma+it)| \le A_1 \log T,$$

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$$|\log\zeta(2+iT)| \leq A_2.$$

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and in particular

$$|\log \zeta(2 - \tfrac{\delta}{4} + iT)| \leq 7(A_1 \log T + A_2).$$

Now, we apply Borel-Carathéodory theorem to the circles $\mathcal{C}_1$ and Γ_1 we have

$$\mathcal{M}_1 \leq 7(A_1 \log T + |\log \zeta(2 - \tfrac{\delta}{4} + iT)|) \leq (7 + 7^2) A_1 \log T + 7^2 A_2.$$

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$$(M_2)^{\log(r_3/r_1)} \leq (M_1)^{\log(r_3/r_2)}(M_3)^{\log(r_2/r_1)}.$$

Then, using Hadamard's three-circles theorem we have

$$M_\nu \leq (m_\nu)^a (\mathcal{M}_\nu)^b,$$

where $a = \log(3/2)/\log 3$, $b = \log 2/\log 3$ and $a + b = 1$.

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We continue until

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Recall that M_0 is bounded (as $T \to \infty$). Then

$$M_n \le A_4\, 7^n\, (\log T)^{1-a^n}.$$

We want a lower bound for M_n. Recall that $M_n \geq \log|\zeta(s)|$ at the circle C_n. We need a lower bound for $|\zeta(s)|$ on C_n ($\mathrm{Re}\, s \leq -1$).

Recalling the functional equation:

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Then, we write

$$\zeta(s) = \chi(s)\,\zeta(1-s),$$

where

$$\chi(s) = \frac{\pi^{-(1-2s)/2}\,\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Using Stiriing's formula we have for $\alpha \le \sigma \le \beta$, as $t \to \infty$:

$$|\chi(s)| \sim \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}}.$$

Then, in $-2 \leq \sigma \leq -1$, we have

$$|\chi(s)| \geq K\left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}} \geq K_1 t^{3/2} \geq K_2 T^{3/2}$$

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and $|\zeta(1-s)| \geq K_3$ (use Mobiüs function).

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and $|\zeta(1-s)| \ge K_3$ (use Mobiüs function). Therefore

$$|\zeta(s)| \ge T^{A_5},$$

in the circle C_n. Therefore $M_n \ge A_5 \log T$.

We have prove that

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Recall that $n = [12/\delta] + 1$.

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Recall that $n = [12/\delta] + 1$. We conclude.