c_ν ; C_ν ; $\mathbf{C}_\nu$; Γ_ν ; $\boxed{\delta<1/2}$

center: $2-\frac{\nu\delta}{4}+iT$

$c_\nu \to \delta/4$

$C_\nu \to \delta/2$ $\quad \nu=0,1,2,\dots n$

$\mathbf{C}_\nu \to 3\delta/4$ $\quad \boxed{2-\frac{n\delta}{4}\le -1}$

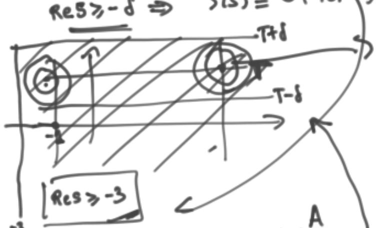
$\Gamma_\nu \to \delta$

$m_\nu = \max_{s\in c_\nu} |\log\zeta(s)|$ $\quad n=[\![\frac{12}{\delta}]\!]+1$

$M_\nu = \max_{s\in C_\nu} |\log\zeta(s)|$

$\mathbf{M}_\nu = \max_{s\in \mathbf{C}_\nu} |\log\zeta(s)|$

$\operatorname{Re} s \ge -\delta \Rightarrow \zeta(s) = O(|t|^{A_\delta})$



$|\zeta(\sigma+it)| \le M|t|^A$

$\boxed{|\ln|\zeta(\sigma+it)|| \le C\log t} \le C\log(T+\delta) \le C\log(T^2) \le C\log T$

$T-\delta \le t \le T+\delta$

$\boxed{|\ln|\zeta(s)|| \le C.\log T}$

$\boxed{\operatorname{Re}\{\log\zeta(s)\} \le A_1 \log T \text{ on the circles}}$

$$|\log\zeta(2+iT)| = \left|\sum \frac{\Lambda(n)}{n^{2+iT}\log n}\right| \le \sum_n \frac{\Lambda(n)}{n^2\log n} = A_2$$

$$\boxed{\operatorname{Re}\{\log\zeta(s)\} \le A_1 . \log T \text{ for all } s \text{ on the circles}} \quad \underline{(*)}$$

$$|\log\zeta(2+iT)| \le A_2 \quad \underline{(**)}$$

(Borel – Carathéodory)

$\mathbf{C}_0,\ \Gamma_0 \to \delta$

$\hookrightarrow s$

center $2+iT$

$\hookrightarrow 3\delta/4$

$$\max_{s\in \mathbf{C}_0} |\log\zeta(s)| \le \frac{2.(3\delta/4)}{\delta-3\delta/4} \cdot \max_{s\in C_0} \operatorname{Re}\{\log\zeta(s)\} + \frac{\frac{3\delta}{4}+\delta}{\delta-\frac{3\delta}{4}} \cdot |\log\zeta(2+iT)|$$

$$\max_{s\in C_0} |\log\zeta(s)| \le \frac{\cancel{6\delta/4}}{\cancel{\delta/4}} \cdot (A_1\log T) + \frac{\cancel{7\delta/4}}{\cancel{\delta/4}} \cdot (A_2)$$

$$\Rightarrow \max_{s\in C_0} |\log\zeta(s)| \le 6A_1\log T + 7A_2 \le 7(A_1\log T + A_2)$$

$$\max_{s\in C_0} |\log \zeta(s)| \le 7(A_1\log T + A_2)$$

$$\boxed{M_0 \le 7(A_1 \log T + A_2)}$$ in particular

$$\left|\log \zeta\left(2-\frac{\delta}{4}+iT\right)\right| \le 7(A_1\log T + A_2)$$

Borel – Carathéodory theorem: C_1 ; Γ_1

$$\max_{s\in C_1}|\log\zeta(s)| \le \frac{2\cdot(3\delta/4)}{\delta-\frac{3\delta}{4}}\cdot \max_{s\in\Gamma_1}\mathrm{Re}\{\log\zeta(s)\} + \frac{\delta+3\delta/4}{\delta-3\delta/4}\cdot \left|\log\zeta\left(2-\frac{\delta}{4}+iT\right)\right|$$

$$M_1 \le 7(A_1\log T) + 7\left(7(A_1 \log T + A_2)\right)$$

$$\boxed{M_1 \le (7+7^2)A_1\log T + 7^2 A_2}$$

$$M_\nu \le (7+7^2+\dots+7^{\nu+1})A_1\log T + 7^{\nu+1}A_2$$

$$M_\nu \le 7(1+7+\dots+7^{\nu})A_1\log T + 7^{\nu+1}A_2$$

$$= 7\left(\frac{7^{\nu+1}-1}{7-1}\right)A_1\log T + 7^{\nu+1}A_2$$

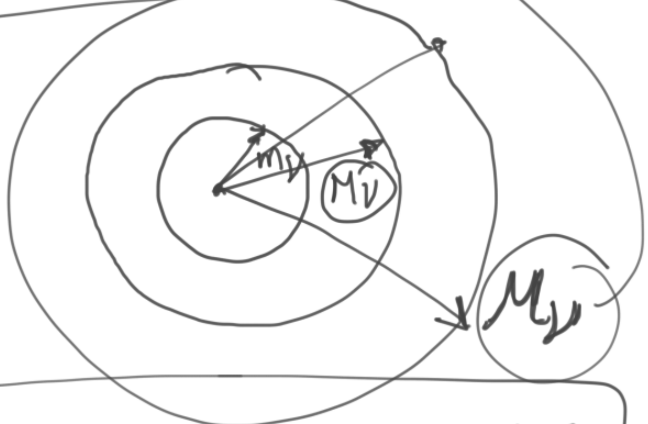
$$\le \frac{7}{6}(7^{\nu+1})A_1\log T + 7^{\nu+1}A_2$$

$$= 7^{\nu}\left(\frac{49}{6}A_1\log T + \underbrace{7A_2}_{7A_2\ \log T}\right)$$

$$\boxed{M_\nu \le 7^{\nu}\cdot A_1\log T}$$

$$\boxed{M_\nu \le 7^{\nu}\cdot A_1\log T}$$

$$\boxed{\mathcal{M}_\nu \le 7^\nu A_1 \log T}$$



$$\frac{3\delta/4}{\delta/4}$$

$$\frac{3\delta/4}{\delta/2} = 3/2$$

$$\frac{\delta/2}{\delta/4} = 2$$

$$\boxed{(\mathcal{M}_\nu)^{\log(3)} \le (m_\nu)^{\log(3/2)} \cdot (M_\nu)^{\log 2}}$$

Hadamard 3-circles Theorem

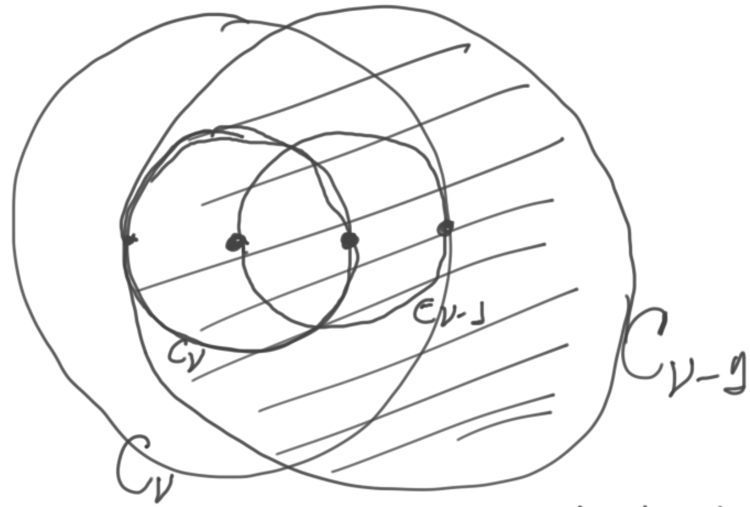
$$\mathcal{M}_\nu \le (m_\nu)^{\log(3/2)/\log 3} \cdot (M_\nu)^{\log 2/\log 3}$$

$a = \log(3/2)/\log 3$

$b = \log 2/\log 3$

$$\boxed{a+b=1}$$

$$\mathcal{M}_\nu \le (m_\nu)^a \cdot (M_\nu)^b$$



Maximum principle Theorem

$$\max_{s \in C_\nu} |\log \varphi(s)| \le \max_{s \in C_{\nu-1}} |\log \varphi(s)|$$

$$\boxed{m_\nu \le \mathcal{M}_{\nu-1}} \qquad \nu \ge 1$$

$$\mathcal{M}_\nu \le (m_\nu)^a \cdot (\mathbb{M}_\nu)^b \quad \text{(Hadamard)}$$

$$\boxed{m_\nu \le \mathcal{M}_{\nu-1}} \quad \text{(inclusion)}$$

$$\mathcal{M}_\nu \le (\mathcal{M}_{\nu-1})^a (\mathbb{M}_\nu)^b, \quad \nu = 1, 2, \dots n$$

$$\boxed{\mathcal{M}_1 \le (\mathcal{M}_0)^a (\mathbb{M}_1)^b}$$

$$\mathcal{M}_2 \le (\mathcal{M}_1)^a (\mathbb{M}_2)^b$$

$$\le \left((\mathcal{M}_0)^a (\mathbb{M}_1)^b\right)^a \cdot (\mathbb{M}_2)^b$$

$$= (\mathcal{M}_0)^{a^2} (\mathbb{M}_1)^{ab} (\mathbb{M}_2)^b$$

$$\Rightarrow \boxed{\mathcal{M}_2 \le (\mathcal{M}_0)^{a^2} (\mathbb{M}_1)^{ab} (\mathbb{M}_2)^b}$$

$$\mathcal{M}_n \le (\mathcal{M}_0)^{a^n} (\mathbb{M}_1)^{a^{n-1}b} (\mathbb{M}_2)^{a^{n-2}b} \cdots (\mathbb{M}_n)^b$$

$$\mathbb{M}_\nu \le 7^\nu A_1 \log T \quad \text{(we have proved)}$$

$$\mathcal{M}_n \le (\mathcal{M}_0)^{a^n} (7^1 A_1 \log T)^{a^{n-1}b} \times (7^2 A_1 \log T)^{a^{n-2}b} \times \cdots \times (7^n A_1 \log T)^b$$

$$\boxed{\mathcal{M}_n \le (\mathcal{M}_0)^{a^n} \cdot 7^{a^{n-1}b + 2a^{n-2}b + \dots + nb} \times (A_1 \log T)^{a^{n-1}b + a^{n-2}b + \dots b}}$$