Class 21: Unconditional gaps and Zeros on the critical line

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Gaps between zeros of zeta

Let $0 < \gamma_1 \leq \gamma_2 \leq \ldots$ be the consecutive ordinates of the non-trivial zeros of $\zeta(s)$ with positive imaginary parts (counting multiplicity).

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2. Unconditional: for some $A > 0$ we have
$$\gamma_{n+1} - \gamma_n \leq \frac{A}{\log\log\log\gamma_n}.$$

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3. Conditional:
$$\gamma_{n+1} - \gamma_n \leq \frac{\pi + \varepsilon}{\log\log \gamma_n}.$$

Theorem (Borel-Carathéodory theorem)

Let $\Omega \subset \mathbb{C}$ be an open set such that contains the disc $|z| \leq R$. Then, for $0 < r < R$ we have that

$$\max_{|z|\leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z|=R} \mathrm{Re}\, f(z) + \frac{R+r}{R-r} |f(0)|.$$

Theorem (Hadamard's three-circles theorem)

Let $\Omega \subset \mathbb{C}$ be an open set such that contains the annulus $r_1 \leq |z| \leq r_3$. Let $f : \Omega \to \mathbb{C}$ be an holomorphic function. Define $M(r)$ the maximum of $|f(z)|$ on the circle $|z| = r$. Then, for $r_1 < r_2 < r_3$ we have:

$$(M_2)^{\log(r_3/r_1)} \leq (M_1)^{\log(r_3/r_2)}(M_3)^{\log(r_2/r_1)}.$$

Theorem

Let T sufficiently large. Then, there is a zero $\rho = \beta + i\gamma$ of $\zeta(s)$ such that

$$|\gamma - T| \leq \frac{A}{\log \log \log T},$$

for some universal constant $A > 0$.

Suppose that $\zeta(s)$ has no zeros in $T-\delta \leq \mathrm{Im}\, s \leq T+\delta$, with $0<\delta<\frac{1}{2}$. Define the function $\log \zeta(s)$, analytic in $-2 \leq \mathrm{Re}\, s \leq 3$ and $T-\delta \leq \mathrm{Im}\, s \leq T+\delta$.

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Let c_ν, C_ν, $\mathcal{C}_\nu$ and Γ_ν be four concentric circles, with centre $2-\frac{\nu}{4}\delta+iT$ and radii $\frac{\delta}{4}$, $\frac{\delta}{2}$, $\frac{3\delta}{4}$ and δ respectively. Consider these set of circles such for $\nu=0,1,...,n$ where $n=[12/\delta]+1$, so that $2-\frac{n}{4}\delta \leq -1$.

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Define m_ν, M_ν and $\mathcal{M}_\nu$ the maxima of $|\log\zeta(s)|$ on the circles c_ν, C_ν, and $\mathcal{C}_\nu$ respectively.

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We have for all circles that

$$\mathrm{Re}\,\{\log\zeta(s)\} = \log|\zeta(\sigma+it)| \le A_1 \log T,$$

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$$\mathrm{Re}\,\{\log\zeta(s)\} = \log|\zeta(\sigma+it)| \le A_1\log T,$$

and

$$|\log\zeta(2+iT)|\le A_2.$$

Theorem (Borel-Carathéodory theorem)

Let $\Omega \subset \mathbb{C}$ be an open set such that contains the disc $|z| \leq R$. Then, for $0 < r < R$ we have that

$$\max_{|z|\leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z|=R} \mathrm{Re}\, f(z) + \frac{R+r}{R-r} |f(0)|.$$

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and in particular

$$|\log \zeta(2 - \tfrac{\delta}{4} + iT)| \le 7(A_1 \log T + A_2).$$

Now, we apply Borel-Carathéodory theorem to the circles $\mathcal{C}_1$ and Γ_1 we have

$$\mathcal{M}_1 \leq 7(A_1 \log T + |\log \zeta(2 - \frac{\delta}{4} + iT)|) \leq (7 + 7^2)A_1 \log T + 7^2 A_2.$$

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In general

$$\mathcal{M}_\nu \leq (7 + 7^2 + ... + 7^{\nu+1})A_1 \log T + 7^{\nu+1} A_2.$$

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$$\mathcal{M}_\nu \leq 7^\nu A_3 \log T.$$

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Let $\Omega \subset \mathbb{C}$ be an open set such that contains the annulus $r_1 \leq |z| \leq r_3$. Let $f : \Omega \to \mathbb{C}$ be an holomorphic function. Define $M(r)$ the maximum of $|f(z)|$ on the circle $|z| = r$. Then, for $r_1 < r_2 < r_3$ we have:

$$(M_2)^{\log(r_3/r_1)} \leq (M_1)^{\log(r_3/r_2)}(M_3)^{\log(r_2/r_1)}.$$

Then, using Hadamard's three-circles theorem we have

$$M_\nu \le (m_\nu)^a (\mathcal{M}_\nu)^b,$$

where $a = \log(3/2)/\log 3$, $b = \log 2/\log 3$ and $a + b = 1$.

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We continue until

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Recall that M_0 is bounded (as $T \to \infty$). Then

$$M_n \leq A_4 7^n (\log T)^{1-a^n}.$$

We want a lower bound for M_n. Recall that $M_n \geq \log|\zeta(s)|$ at the circle C_n. We need a lower bound for $|\zeta(s)|$ on C_n $(\mathrm{Re}\, s \leq -1)$.

Recalling the functional equation:

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Then, we write

$$\zeta(s) = \chi(s)\,\zeta(1-s),$$

where

$$\chi(s) = \frac{\pi^{-(1-2s)/2}\,\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Using Stiriing's formula we have for $\alpha \le \sigma \le \beta$, as $t \to \infty$:

$$|\chi(s)| \sim \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}}.$$

Then, in $-2 \le \sigma \le -0.5$, we have

$$|\chi(s)| \ge K\left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}} \ge K_1 t^C \ge K_2 T^C$$

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and $|\zeta(1-s)| \geq K_3$ (use Mobiüs function). Therefore

$$|\zeta(s)| \geq T^{A_5},$$

in the circle C_n. Therefore $M_n \geq A_5 \log T$.

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Recall that $n = [12/\delta] + 1$.

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Recall that $n = [12/\delta] + 1$. We conclude.

Let $N(T) = \#\{\rho = \beta + i\gamma : 0 < \beta < 1, \zeta(\rho) = 0, 0 < \gamma \leq T\}$.

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For $T \geq 3$ we have the Riemann-von Mangoldt Formula:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right).$$

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Since $S(T) = O(\log T)$, we get

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} + O(T).$$

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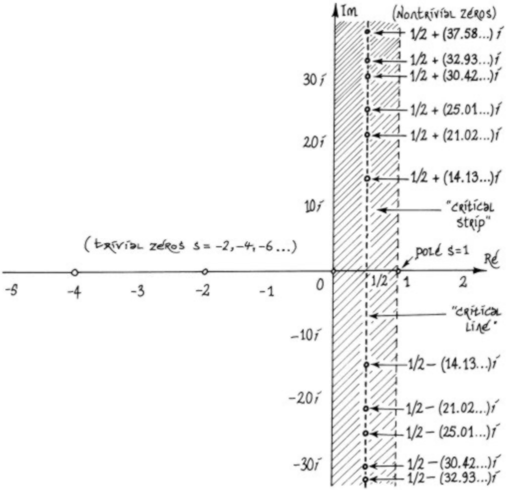
3. Density conjecture: $N(\sigma, T) \ll T^{2(1-\sigma)} \log T$.
4. Selberg : $N(\sigma, T) \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})} \log T$.



Im
(Nontrivial zeros)
1/2 + (37.58...) i
1/2 + (32.93...) i
1/2 + (30.42...) i
1/2 + (25.01...) i
1/2 + (21.02...) i
1/2 + (14.13...) i
"critical strip"
(trivial zeros s = -2, -4, -6 ...)
pole s=1
Re
-5 -4 -3 -2 -1 0 1/2 1 2
30 i
20 i
10 i
-10 i
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"critical line"
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1. Riemann hypothesis is equivalent to $N_0(T) = N(T)$.
2. Dave Platt and Tim Trudgian, 21 April 2020:
   The Riemann Hypothesis is true up to height 3000175332800.
   That is, the lowest 12363153437138 non-trivial zeros ρ have $\mathrm{Re}\,\rho = 1/2$.

Theorem (Hardy and Littlewood: 1921)

For $T \geq 15$ *we have*

$$N_0(T) \gg T.$$

Define

$$Z(u) = \frac{H(\frac{1}{2}+iu)}{|H(\frac{1}{2}+iu)|}\zeta(\tfrac{1}{2}+iu),$$

where

$$H(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

Recall the functional equation $H(s)\zeta(s) = H(1-s)\zeta(1-s)$. Note that $Z(u)$ is real an even, for $u \in \mathbb{R}$. Then Z changes sign when ζ has a zero on the critical line. Therefore we want to show that the sign change of $Z(u)$ occurs quite often.

Let us compare the integrals:

$$I(t) = \int_t^{t+\Delta} Z(u)\,\mathrm{d}u,$$

and

$$J(t) = \int_t^{t+\Delta} |Z(u)|\,\mathrm{d}u,$$

in the range $T \leq t \leq 2T$ and Δ large to be chosen later ($\Delta \leq T^{1/6}$).

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in the range $T \leq t \leq 2T$ and Δ large to be chosen later ($\Delta \leq T^{1/6}$). We need an upper bound for $|I(t)|$ and a lower bound for $J(t)$ on average over a subset $\mathcal{T} \subset [T, 2T]$.

We remark that, for $s = \frac{1}{2} + iu$ with $T \leq u \leq 3T$:

$$\zeta\left(\frac{1}{2} + iu\right) = \sum_{n \leq T} n^{-\frac{1}{2} - iu} + O(T^{-1/2}).$$

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Then

$$\begin{aligned} J(t) = \int_t^{t+\Delta} \left|\zeta(\tfrac{1}{2} + iu)\right| \mathrm{d}u &\geq \left| \int_t^{t+\Delta} \zeta(\tfrac{1}{2} + iu)\, \mathrm{d}u \right| \\ &\geq \Delta - \left| \int_t^{t+\Delta} \left(\zeta(\tfrac{1}{2} + iu) - 1\right) \mathrm{d}u \right| \\ &\geq \Delta - \left| \int_t^{t+\Delta} \left(\sum_{1 < n \leq T} n^{-\frac{1}{2} - iu} \right) \mathrm{d}u \right| + O(\Delta T^{-1/2}) \\ &\geq \Delta - \left| \sum_{1 < n \leq T} \frac{1 - n^{-i\Delta}}{\log n} n^{-\frac{1}{2} - it} \right| + O(\Delta T^{-1/2}). \end{aligned}$$

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$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 \mathrm{d}t = (T + O(N)) \sum_{n=1}^{N} |a_n|^2.$$

This implies

$$\int_T^{2T} \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right|^2 \mathrm{d}t \ll T.$$

Therefore, from

$$J(t) \geq \Delta - \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right| + O(\Delta T^{-1/2}),$$

and

$$\int_T^{2T} \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right|^2 \mathrm{d}t \ll T,$$

we obtain, for any subset $\mathcal{T} \subset [T, 2T]$, we have

$$\int_{\mathcal{T}} J(t)\mathrm{d}t > \Delta|\mathcal{T}| + O\big(|\mathcal{T}|^{1/2} T^{1/2} + \Delta|\mathcal{T}| T^{-1/2}\big).$$