Class 22: Zeros on the critical line II

Andrés Chirre
Norwegian University of Science and Technology - NTNU

18-November-2021

Theorem (Hardy and Littlewood: 1921)

For $T \geq 15$ we have

$$N_0(T) \gg T,$$

where $N_0(T)$ is the number of zeros $\rho = \frac{1}{2} + i\gamma$ with $0 < \gamma \leq T$.

Define

$$Z(u) = \frac{H(\frac{1}{2}+iu)}{|H(\frac{1}{2}+iu)|}\zeta(\tfrac{1}{2}+iu),$$

where

$$H(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

Recall the functional equation $H(s)\zeta(s) = H(1-s)\zeta(1-s)$. Note that $Z(u)$ is real an even, for $u \in \mathbb{R}$. Then, if Z changes sign, ζ has a zero on the critical line. Therefore we want to show that the sign change of $Z(u)$ occurs quite often.

Let T sufficiently large. Let us define

$$I(t) = \int_t^{t+\Delta} Z(u)\,\mathrm{d}u,$$

and

$$J(t) = \int_t^{t+\Delta} |Z(u)|\,\mathrm{d}u,$$

in the range $T \leq t \leq 2T$ and $1 \leq \Delta \leq T^{1/6}$ large to be chosen.

Let T sufficiently large. Let us define

$$I(t) = \int_t^{t+\Delta} Z(u)\,\mathrm{d}u,$$

and

$$J(t) = \int_t^{t+\Delta} |Z(u)|\,\mathrm{d}u,$$

in the range $T \leq t \leq 2T$ and $1 \leq \Delta \leq T^{1/6}$ large to be chosen.

We need a lower bound for $\int_{\mathcal{T}} J(t)\,\mathrm{d}t$, and an upper bound for $\int_{\mathcal{T}} |I(t)|\,\mathrm{d}t$ over a subset $\mathcal{T} \subset [T, 2T]$.

We remark that, for $s = \frac{1}{2} + iu$ with $T \leq u \leq 3T$:

$$\zeta\left(\frac{1}{2} + iu\right) = \sum_{n \leq T} n^{-\frac{1}{2} - iu} + O(T^{-1/2}).$$

We remark that, for $s = \frac{1}{2} + iu$ with $T \leq u \leq 3T$:

$$\zeta\left(\frac{1}{2} + iu\right) = \sum_{n \leq T} n^{-\frac{1}{2} - iu} + O(T^{-1/2}).$$

Then

$$\begin{aligned} J(t) = \int_t^{t+\Delta} \left|\zeta(\tfrac{1}{2} + iu)\right| \mathrm{d}u &\geq \left| \int_t^{t+\Delta} \zeta(\tfrac{1}{2} + iu) \, \mathrm{d}u \right| \\ &\geq \Delta - \left| \int_t^{t+\Delta} \left(\zeta(\tfrac{1}{2} + iu) - 1\right) \mathrm{d}u \right| \\ &\geq \Delta - \left| \int_t^{t+\Delta} \left(\sum_{1 < n \leq T} n^{-\frac{1}{2} - iu} \right) \mathrm{d}u \right| + O(\Delta T^{-1/2}) \\ &\geq \Delta - \left| \sum_{1 < n \leq T} \frac{1 - n^{-i\Delta}}{\log n} n^{-\frac{1}{2} - it} \right| + O(\Delta T^{-1/2}). \end{aligned}$$

To bound the term

$$\left| \sum_{1<n\leq T} \frac{1 - n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right|,$$

we use Problem (50):

To bound the term

$$\left| \sum_{1<n\le T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right|,$$

we use Problem (50): Let $\{a_n\}_{n=1}^{N}$ be complex numbers. Then, we have for $T \geq 2$:

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{it} \right|^2 \mathrm{d}t = (T + O(N)) \sum_{n=1}^{N} |a_n|^2.$$

To bound the term

$$\left|\sum_{1<n\leq T}\frac{1-n^{-i\Delta}}{\log n}n^{-\frac{1}{2}-it}\right|,$$

we use Problem (50): Let $\{a_n\}_{n=1}^{N}$ be complex numbers. Then, we have for $T\geq 2$:

$$\int_0^T\left|\sum_{n=1}^{N}a_n n^{it}\right|^2\mathrm{d}t=(T+O(N))\sum_{n=1}^{N}|a_n|^2.$$

This implies

$$\int_T^{2T}\left|\sum_{1<n\leq T}\frac{1-n^{-i\Delta}}{\log n}n^{-\frac{1}{2}-it}\right|^2\mathrm{d}t\ll T.$$

Therefore, from

$$J(t) \geq \Delta - \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right| + O(\Delta T^{-1/2}),$$

and

$$\int_T^{2T} \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right|^2 \mathrm{d}t \ll T,$$

Therefore, from

$$J(t) \geq \Delta - \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right| + O(\Delta T^{-1/2}),$$

and

$$\int_T^{2T} \left| \sum_{1<n\leq T} \frac{1-n^{-i\Delta}}{\log n} n^{-\frac{1}{2}-it} \right|^2 \mathrm{d}t \ll T,$$

using Cauchy-Schwarz inequality we obtain, for any subset $\mathcal{T} \subset [T, 2T]$,

$$\int_{\mathcal{T}} J(t)\, \mathrm{d}t > \Delta|\mathcal{T}| + O\big(|\mathcal{T}|^{1/2} T^{1/2} + \Delta|\mathcal{T}| T^{-1/2}\big).$$

Lemma

For $1 \le \Delta \le T^{1/6}$ *we have:*

$$\int_T^{2T} |I(t)|^2 \mathrm{d}t \ll \Delta T.$$

Lemma

For $1 \leq \Delta \leq T^{1/6}$ *we have:*

$$\int_T^{2T} |I(t)|^2 \mathrm{d}t \ll \Delta T.$$

Assume this lemma, and let us prove our theorem.

Using Cauchy-Schwarz inequality we have for any subset $\mathcal{T} \subset [T, 2T]$:

$$\int_{\mathcal{T}} |I(t)| \, \mathrm{d}t \leq \left(\int_{\mathcal{T}} |I(t)|^2 \, \mathrm{d}t \right)^{1/2} \left(\int_{\mathcal{T}} 1 \, \mathrm{d}t \right)^{1/2} \ll \Delta^{1/2} |\mathcal{T}|^{1/2} T^{1/2}.$$

Let T sufficiently large and $1 \le \Delta \le T^{1/6}$. For

$$I(t) = \int_t^{t+\Delta} Z(u)\,\mathrm{d}u, \quad \text{and} \quad J(t) = \int_t^{t+\Delta} |Z(u)|\,\mathrm{d}u,$$

we have that:

$$\int_{\mathcal{T}} J(t)\,\mathrm{d}t > \Delta|\mathcal{T}| + O\big(|\mathcal{T}|^{1/2}T^{1/2} + \Delta|\mathcal{T}|T^{-1/2}\big),$$

and

$$\int_{\mathcal{T}} |I(t)|\,\mathrm{d}t \ll \Delta^{1/2}|\mathcal{T}|^{1/2}T^{1/2}.$$

Now, consider $\mathcal{T}$ the subset of $[T, 2T]$ such that: $|I(t)| = J(t)$. This is the set of t's such that $Z(u)$ does not chang sign in the interval $(t, t + \Delta)$.

Now, consider $\mathcal{T}$ the subset of $[T, 2T]$ such that: $|I(t)| = J(t)$. This is the set of t's such that $Z(u)$ does not chang sign in the interval $(t, t+\Delta)$. Then

$$\int_{\mathcal{T}} |I(t)| \, \mathrm{d}t = \int_{\mathcal{T}} J(t) \, \mathrm{d}t,$$

Now, consider $\mathcal{T}$ the subset of $[T, 2T]$ such that: $|I(t)| = J(t)$. This is the set of t's such that $Z(u)$ does not chang sign in the interval $(t, t + \Delta)$. Then

$$\int_{\mathcal{T}} |I(t)| \, \mathrm{d}t = \int_{\mathcal{T}} J(t) \, \mathrm{d}t,$$

and we deduce that $|\mathcal{T}| \ll \Delta^{-1} T$.

Now, consider $\mathcal{T}$ the subset of $[T, 2T]$ such that: $|I(t)| = J(t)$. This is the set of t's such that $Z(u)$ does not chang sign in the interval $(t, t+\Delta)$. Then

$$\int_{\mathcal{T}} |I(t)| \, \mathrm{d}t = \int_{\mathcal{T}} J(t) \, \mathrm{d}t,$$

and we deduce that $|\mathcal{T}| \ll \Delta^{-1} T$. Then, we choose Δ sufficiently large such that

$$|\mathcal{T}| \leq \frac{T}{2}.$$

Since

$$|\mathcal{T}| \leq \frac{T}{2},$$

Since

$$|\mathcal{T}| \le \frac{T}{2},$$

the set $\mathcal{S} = [T, 2T] - \mathcal{T}$, that is that set of t's such that $|I(t)| < J(t)$ has measure

$$|\mathcal{S}| > \frac{T}{2}.$$

Since

$$|\mathcal{T}| \leq \frac{T}{2},$$

the set $\mathcal{S} = [T, 2T] - \mathcal{T}$, that is that set of t's such that $|I(t)| < J(t)$ has measure

$$|\mathcal{S}| > \frac{T}{2}.$$

The set $\mathcal{S}$ contains a sequence $\{t_1, ..., t_R\}$ of Δ-spaced points of lenght $R \geq T/4\Delta$. For every t_r, the function $Z(u)$ must change sign in the segment $t_r < u < t_r + \Delta$, hence there is a critical zero $\rho = \frac{1}{2} + i\gamma_r$ with $t_r < \gamma_r < t_r + \Delta$.

Since

$$|\mathcal{T}| \leq \frac{T}{2},$$

the set $\mathcal{S} = [T, 2T] - \mathcal{T}$, that is that set of t's such that $|I(t)| < J(t)$ has measure

$$|\mathcal{S}| > \frac{T}{2}.$$

The set $\mathcal{S}$ contains a sequence $\{t_1, ..., t_R\}$ of Δ-spaced points of lenght $R \geq T/4\Delta$. For every t_r, the function $Z(u)$ must change sign in the segment $t_r < u < t_r + \Delta$, hence there is a critical zero $\rho = \frac{1}{2} + i\gamma_r$ with $t_r < \gamma_r < t_r + \Delta$.

Therefore the number of critical zeros $\rho = \frac{1}{2} + i\gamma$ with $T < \gamma < 2T$ is at least $T/4\Delta - 1$. We conclude.

Proof of Lemma

By the convexity bound we have that

$$\begin{aligned}
&\int_T^{2T} |I(t)|^2 \mathrm{d}t \\
&\quad = \int_T^{2T} \left| \int_0^{\Delta} Z(t+u)\,\mathrm{d}u \right|^2 \mathrm{d}t \\
&\quad = \int_0^{\Delta}\int_0^{\Delta}\int_T^{2T} Z(t+u_1)\overline{Z(t+u_2)}\,\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2 \\
&\quad = \int_0^{\Delta}\int_0^{\Delta}\int_T^{2T} Z(t)\overline{Z(t+u_2-u_1)}\,\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2 + O\big(\Delta^3 T^{1/2}\big) \\
&\quad = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t\,\mathrm{d}u + O\big(\Delta^3 T^{1/2}\big).
\end{aligned}$$

Using Stirling's formula, we have that

$$\frac{H(\frac{1}{2}+it)\overline{H(\frac{1}{2}+it+iu)}}{\left|H(\frac{1}{2}+it)\overline{H(\frac{1}{2}+it+iu)}\right|} = \left(\frac{2\pi}{t}\right)^{iu/2}\left(1+O\left(\frac{u^2+1}{T}\right)\right),$$

for $T \le t \le 2T$ and $|u| \le \Delta$.

Using Stirling's formula, we have that

$$\frac{H(\frac{1}{2}+it)\overline{H(\frac{1}{2}+it+iu)}}{\left|H(\frac{1}{2}+it)\overline{H(\frac{1}{2}+it+iu)}\right|} = \left(\frac{2\pi}{t}\right)^{iu/2}\left(1+O\left(\frac{u^2+1}{T}\right)\right),$$

for $T \le t \le 2T$ and $|u| \le \Delta$. Then, using the convexity bound and the approximation formula we have

$$\begin{aligned}
Z(t)\overline{Z(t+u)} &= \zeta(\tfrac{1}{2}+it)\overline{\zeta(\tfrac{1}{2}+it+iu)}\left(\frac{2\pi}{t}\right)^{iu/2} + O(\Delta^2 T^{-1/2}) \\
&= \sum_{1\le m,n\le T} \frac{1}{\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\left(\frac{2\pi m^2}{t}\right)^{iu/2} + O(\Delta^2 T^{-1/2}) \\
&\quad + O\left(T^{-1/2}\left|\sum_{n\le T}\frac{1}{n^{1/2+it}}\right|\right) + O\left(T^{-1/2}\left|\sum_{m\le T}\frac{1}{m^{1/2-it-iu}}\right|\right)
\end{aligned}$$

$$\int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t = \int_T^{2T} \sum_{1\le m,n\le T} \frac{1}{\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\left(\frac{2\pi m^2}{t}\right)^{iu/2}\mathrm{d}t$$

$$+ O(\Delta^2 T^{1/2})$$

$$+O\left(T^{-1/2}\int_T^{2T}\left|\sum_{n\le T}\frac{1}{n^{1/2+it}}\right|\mathrm{d}t\right)$$

$$+O\left(T^{-1/2}\int_T^{2T}\left|\sum_{m\le T}\frac{1}{m^{1/2-it-iu}}\right|\mathrm{d}t\right).$$

$$\int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t = \int_T^{2T} \sum_{1\leq m,n\leq T} \frac{1}{\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\left(\frac{2\pi m^2}{t}\right)^{iu/2}\mathrm{d}t$$

$$+ O(\Delta^2 T^{1/2}) + O((\log T)^{1/2} T^{1/2}).$$

$$
\begin{aligned}
\int_T^{2T} |I(t)|^2 \mathrm{d}t \\
&= \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t\,\mathrm{d}u + O\big(\Delta^3 T^{1/2}\big) \\
&= \sum_{1 \le m,n \le T} \frac{c(m,n)}{\sqrt{mn}} + O(\Delta^2 (\log T)^{1/2} T^{1/2}) + O(\Delta^4 T^{1/2}),
\end{aligned}
$$

$$\int_T^{2T} |I(t)|^2 \mathrm{d}t$$

$$= \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t\,\mathrm{d}u + O\big(\Delta^3 T^{1/2}\big)$$

$$= \sum_{1 \le m,n \le T} \frac{c(m,n)}{\sqrt{mn}} + O(\Delta^2 (\log T)^{1/2} T^{1/2}) + O(\Delta^4 T^{1/2}),$$

where

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t\,\mathrm{d}u.$$

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_{T}^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t \, \mathrm{d}u$$

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t \, \mathrm{d}u$$

Then

$$c(m,n) = \Delta^2 \int_T^{2T} \left(\frac{m}{n}\right)^{it} F\left(\frac{\Delta}{4} \log \frac{2\pi m^2}{t}\right),$$

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t \, \mathrm{d}u$$

Then

$$c(m,n) = \Delta^2 \int_T^{2T} \left(\frac{m}{n}\right)^{it} F\left(\frac{\Delta}{4} \log \frac{2\pi m^2}{t}\right),$$

where

$$F(u) = \left(\frac{\sin x}{x}\right)^2.$$

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_{T}^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t \, \mathrm{d}u$$

Then

$$c(m,n) = \Delta^2 \int_{T}^{2T} \left(\frac{m}{n}\right)^{it} F\left(\frac{\Delta}{4} \log \frac{2\pi m^2}{t}\right),$$

where

$$F(u) = \left(\frac{\sin x}{x}\right)^2.$$

To be continue...