Class 23: Zeros on the critical line III

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1. Riemann hypothesis is equivalent to $N_0(T) = N(T)$.
2. Dave Platt and Tim Trudgian, 21 April 2020:
   The Riemann Hypothesis is true up to height 3×10^{12}.

Theorem (Hardy and Littlewood: 1921)

For $T \geq 15$ we have

$$N_0(T) \gg T,$$

where $N_0(T)$ is the number of zeros $\rho = \frac{1}{2} + i\gamma$ with $0 < \gamma \leq T$.

Define

$$Z(u) = \frac{H(\frac{1}{2} + iu)}{|H(\frac{1}{2} + iu)|}\zeta(\tfrac{1}{2} + iu),$$

and

$$I(t) = \int_t^{t+\Delta} Z(u)\,\mathrm{d}u,$$

Define

$$Z(u) = \frac{H(\frac{1}{2}+iu)}{|H(\frac{1}{2}+iu)|}\zeta(\tfrac{1}{2}+iu),$$

and

$$I(t) = \int_t^{t+\Delta} Z(u)\,\mathrm{d}u,$$

Lemma

For $1 \le \Delta \le T^{1/6}$ *we have:*

$$\int_T^{2T} |I(t)|^2 \mathrm{d}t \ll \Delta T.$$

Proof of Lemma

By the convexity bound we have that

$$\begin{aligned}\int_T^{2T} |I(t)|^2 \mathrm{d}t &= \int_T^{2T} \left| \int_0^{\Delta} Z(t+u)\,\mathrm{d}u \right|^2 \mathrm{d}t \\ &= \int_0^{\Delta}\int_0^{\Delta}\int_T^{2T} Z(t+u_1)\overline{Z(t+u_2)}\,\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2 \\ &= \int_0^{\Delta}\int_0^{\Delta}\int_T^{2T} Z(t)\overline{Z(t+u_2-u_1)}\,\mathrm{d}t\,\mathrm{d}u_1\,\mathrm{d}u_2 + O\big(\Delta^3 T^{1/2}\big) \\ &= \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t\,\mathrm{d}u + O\big(\Delta^3 T^{1/2}\big).\end{aligned}$$

Using Stirling's formula, we have that

$$\frac{H(\frac{1}{2}+it)\overline{H(\frac{1}{2}+it+iu)}}{\left|H(\frac{1}{2}+it)\overline{H(\frac{1}{2}+it+iu)}\right|} = \left(\frac{2\pi}{t}\right)^{iu/2}\left(1+O\left(\frac{u^2+1}{T}\right)\right),$$

for $T \leq t \leq 2T$ and $|u| \leq \Delta$.

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for $T\le t\le 2T$ and $|u|\le\Delta$. Then, using the convexity bound and the approximation formula we have

$$\begin{aligned}Z(t)\overline{Z(t+u)}&=\zeta(\tfrac{1}{2}+it)\overline{\zeta(\tfrac{1}{2}+it+iu)}\left(\frac{2\pi}{t}\right)^{iu/2}+O(\Delta^2T^{-1/2})\\&=\sum_{1\le m,n\le T}\frac{1}{\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\left(\frac{2\pi m^2}{t}\right)^{iu/2}+O(\Delta^2T^{-1/2})\\&\quad+O\left(T^{-1/2}\left|\sum_{n\le T}\frac{1}{n^{1/2+it}}\right|\right)+O\left(T^{-1/2}\left|\sum_{m\le T}\frac{1}{m^{1/2-it-iu}}\right|\right)\end{aligned}$$

$$\int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t = \int_T^{2T} \sum_{1\le m,n\le T} \frac{1}{\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\left(\frac{2\pi m^2}{t}\right)^{iu/2}\mathrm{d}t$$
$$+ O(\Delta^2 T^{1/2})$$
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$$\int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t = \int_T^{2T} \sum_{1\le m,n\le T} \frac{1}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t$$
$$+ O(\Delta^2 T^{1/2}) + O((\log T)^{1/2} T^{1/2}).$$

$$\int_T^{2T} |I(t)|^2 \mathrm{d}t$$
$$= \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} Z(t)\overline{Z(t+u)}\,\mathrm{d}t\,\mathrm{d}u + O\big(\Delta^3 T^{1/2}\big)$$
$$= \sum_{1 \le m,n \le T} \frac{c(m,n)}{\sqrt{mn}} + O(\Delta^2 (\log T)^{1/2} T^{1/2}) + O(\Delta^4 T^{1/2}),$$

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where

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_T^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t\,\mathrm{d}u.$$

Then, we should prove that

$$\sum_{1\leq m,n\leq T} \frac{c(m,n)}{\sqrt{mn}} \ll \Delta T.$$

$$c(m,n) = \int_{-\Delta}^{\Delta} (\Delta - |u|) \int_{T}^{2T} \left(\frac{m}{n}\right)^{it} \left(\frac{2\pi m^2}{t}\right)^{iu/2} \mathrm{d}t \, \mathrm{d}u$$

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Then

$$c(m, n) = \Delta^2 \int_{T}^{2T} \left(\frac{m}{n}\right)^{it} F\left(\frac{\Delta}{4} \log \frac{2\pi m^2}{t}\right) \mathrm{d}t,$$

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where

$$F(u) = \left(\frac{\sin x}{x}\right)^2.$$

In the case $m = n$ we have

$$
\begin{aligned}
\sum_{1 \le m \le T} \frac{c(m,m)}{m} &= \sum_{1 \le m \le T} \frac{\Delta^2}{m} \int_T^{2T} F\left(\frac{\Delta}{4} \log \frac{2\pi m^2}{t}\right) \mathrm{d}t \\
&= \Delta^2 \sum_{1 \le m \le T} m \int_{T/m^2}^{2T/m^2} F\left(\frac{\Delta}{4} \log \frac{2\pi}{t}\right) \mathrm{d}t \\
&\le \Delta^2 \int_1^{2T} y \left(\int_{T/y^2}^{8T/y^2} F\left(\frac{\Delta}{4} \log \frac{2\pi}{t}\right) \mathrm{d}t \right) \mathrm{d}y \\
&\ll \Delta T.
\end{aligned}
$$

In the case $m \neq n$, we use integration by parts to obtain the desired result.

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7. 2011: Bui, Conrey and Young: $N_0(T) > \frac{41}{100} N(T)$.

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