In the case $m = n$ we have

$$\sum_{m\leq T} \frac{c(m,m)}{m} = \sum_{m\leq T} \frac{\Delta^2}{m} \int_T^{2T} F\left(\frac{\Delta}{4}\log \frac{2\pi m^2}{t}\right) dt \quad \text{change } \frac{m^2}{t} = \frac{1}{u}$$

$$= \Delta^2 \sum_{m\leq T} m \int_{T/m^2}^{2T/m^2} F\left(\frac{\Delta}{4}\log\frac{2\pi}{u}\right) du \quad \dots (\star)$$

Now, for $1 \leq m \leq T$, take $\quad m \leq y \leq m+1 \quad \dots (1)$

$$\Rightarrow T/y^2 \leq T/m^2 \quad \dots (2)$$

and

$$\frac{2T}{m^2} \leq \frac{8T}{y^2} \quad \dots (3)$$

Then, using (1), (2), (3) and the fact that $F \geq 0$, we have

$$m \int_{T/m^2}^{2T/m^2} F\left(\frac{\Delta}{4}\log\frac{2\pi}{u}\right) du \leq y \int_{T/y^2}^{8T/y^2} F\left(\frac{\Delta}{4}\log\frac{2\pi}{u}\right) du \quad \forall\, y \in [m, m+1]$$

We integrate the above expression (on $[m, m+1]$) and summing from 1 until T we get in $(\star)$
"m"

$$\leq \Delta^2 \int_1^{T+1} y \int_{T/y^2}^{8T/y^2} F\left(\Delta/4 \log\frac{2\pi}{u}\right) du\, dy \leq \Delta^2 \int_1^{2T} y \int_{T/y^2}^{8T/y^2} F\left(\Delta/4 \log\frac{2\pi}{u}\right) du\, dy$$

Now, we use Fubini, as in the class

$$= \Delta^2 \int_1^{2T} y \int_{1/4T}^{8T} \mathbb{1}_{\left[\frac{T}{y^2};\, \frac{8T}{y^2}\right]}(u)\, F\left(\frac{\Delta}{4}\log\frac{2\pi}{u}\right) du\, dy$$

$$= \Delta^2 \int_{1/4T}^{8T} \left(\int_1^{2T} y \cdot \mathbb{1}_{\left[\sqrt{\frac{T}{u}};\, \sqrt{\frac{8T}{u}}\right]}(y)\, dy \right) F\left(\frac{\Delta}{4}\log\frac{2\pi}{u}\right) du$$

We split the above integral in 2 integrals ($\left[1/4T;\, 2/T\right]$ and $\left[\frac{2}{T};\, 8T\right]$

$$I_1 = \Delta^2 \int_{2/T}^{8T} \left(\int_1^{2T} y \cdot 1_{\left[\sqrt{\frac{T}{u}};\, \sqrt{\frac{8T}{u}}\right]}(y)\, dy \right) F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) du$$

Since $\frac{2}{T} \le u \le 8T \Rightarrow \qquad \sqrt{\frac{8T}{u}} \le 2T$

$$\Rightarrow I_1 \le \Delta^2 \int_{2/T}^{8T} \left(\int_1^{\sqrt{\frac{8T}{u}}} y\, dy \right) F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) du \le \Delta^2 \int_{2/T}^{8T} \frac{8T}{2u} F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) du$$

$$= 4\Delta^2 T \int_{2/T}^{8T} F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) \frac{1}{u}\, du =$$

$\xrightarrow{\text{change of variable}}$ $\frac{\Delta}{4} \log \frac{2\pi}{u} = \omega \quad \left(-\frac{\Delta}{4} \frac{du}{u} = d\omega \right)$

$$= 4\Delta^2 T \int_{\frac{\Delta}{4}\log \frac{2\pi}{8T}}^{\Delta/4 \log \pi T} F(\omega)\, 4 \frac{d\omega}{\Delta} = 16 \Delta T \int_{\Delta/4 \log(\ )}^{\Delta/4 \log \pi T} F(\omega)\, d\omega \le 16 \Delta T \underbrace{\int_{-\infty}^{\infty} F(\omega)\, d\omega}_{\text{Constant}}$$

$F(x) = \left(\frac{\sin x}{x}\right)^2$

$$\ll \underline{\Delta T}_{\Delta}$$

$$I_2 = \Delta^2 \int_{1/4T}^{2/T} \left(\underbrace{\int_1^{2T} y \cdot 1_{\left[\sqrt{\frac{T}{u}};\, \sqrt{\frac{8T}{u}}\right]}(y)\, dy}_{} \right) F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) du$$

$$\le \Delta^2 \int_{1/4T}^{2/T} (4T^2)\, F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) du = 4\Delta^2 T^2 \int_{1/4T}^{2/T} \underset{u \le 2/T}{u}\, F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) \frac{du}{u}$$

$$= 8\Delta^2 T \int_{1/4T}^{2/T} F\left(\frac{\Delta}{4} \log \frac{2\pi}{u}\right) \frac{du}{u}$$

$\Rightarrow$ change of variable as before $\left(< \frac{C}{\Delta} \right)$

$$\ll \underline{\Delta T}_{\Delta}$$

When $m \neq n$

$$C(m,n) = \Delta^2 \int_T^{2T} \left(\frac{m}{n}\right)^{it} F\left(\frac{\Delta}{4} \log \frac{2\pi m^2}{t}\right) dt \quad . \quad \text{Note that } \frac{d}{dt}\left(\left(\frac{m}{n}\right)^{it}\right) = \left(\frac{m}{n}\right)^{it} i\log(m/n)$$

$$= \frac{\Delta^2}{i\log(m/n)} \int_T^{2T} \frac{d}{dt}\left(\left(\frac{m}{n}\right)^{it}\right) F\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{t}\right) dt \quad \text{integration by parts}$$

$$= \frac{\Delta^2}{i\log(m/n)} \left[\left(\frac{m}{n}\right)^{it} F\left(\Delta/4 \log \frac{2\pi m^2}{t}\right)\Big|_T^{2T} + \int_T^{2T} \left(\frac{m}{n}\right)^{it} F'\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{t}\right) \cdot \frac{\Delta}{4}\frac{dt}{t} \right]$$

$$\Rightarrow \left|\log(m/n)\, C(m,n)\right| \leq \Delta^2 F\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{T}\right) + \Delta^2 F\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{T}\right)$$
$$+ \Delta^3 \int_T^{2T} \left|F'\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{t}\right)\right| \frac{dt}{t} \quad \cdots (*)$$

Clearly $F(x) = \left(\frac{\sin x}{x}\right)^2 \leq \min\left\{1; \frac{1}{x}\right\}$

$$\Rightarrow \Delta^2 F\left(\frac{\Delta}{4}\log\frac{\pi m^2}{T}\right) + \Delta^2 F\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{T}\right) \leq \min\left\{\Delta^2; 4\Delta\left|\log\frac{\pi m^2}{T}\right|^{-1}\right\} + \min\left\{\Delta^2; 4\Delta\left|\log\frac{2\pi m^2}{T}\right|^{-1}\right\}$$

$$\underset{\text{const}}{\ll} \Delta \underbrace{\min\left\{\Delta; \left|\log\frac{\pi m^2}{T}\right|^{-1} + \left|\log\frac{2\pi m^2}{T}\right|^{-1}\right\}}_{\text{define as } C(m)}$$

$$\Rightarrow \Delta^2 F\left(\frac{\Delta}{4}\log\frac{\pi m^2}{T}\right) + \Delta^2 F\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{T}\right) \ll \Delta\, C(m)$$

The last term is bounded by (Note that $|F'(x)| \ll \min\{1; 1/x^2\}$)

$$\Delta^3 \int_T^{2T} \left|F'\left(\frac{\Delta}{4}\log\frac{2\pi m^2}{t}\right)\right| \frac{dt}{t} = \Delta^3 \int_{\frac{\Delta}{4}\log\frac{2\pi m^2}{2T}}^{\Delta/4 \log\frac{2\pi m^2}{T}} |F'(w)| \frac{4\,dw}{\Delta} = 4\Delta^2 \int_{\Delta/4 \log\left(\frac{\pi m^2}{T}\right)}^{\Delta/4\log\left(\frac{2\pi m^2}{T}\right)} |F'(w)|\, dw$$

$$\ll \Delta^2 \int_{\Delta/4\log\left(\frac{\pi m^2}{T}\right)}^{\Delta/4\log\left(\frac{2\pi m^2}{T}\right)} |F'(w)|\, dw = \Delta^2 \int_{\frac{\Delta}{4}\log\left(\frac{\pi m^2}{T}\right)}^{\frac{\Delta}{4}\log 2 + \frac{\Delta}{4}\log\left(\frac{\pi m^2}{T}\right)} |F'(w)|\, dw \quad \text{using } |F'(x)| \leq \min\left\{1; \frac{1}{x^2}\right\}$$

$$\ll \Delta\, C(m)$$

Therefore, in $(*)$:

$$\left|\log(m/n)\, C(m,n)\right| \ll \Delta\, C(m) \Rightarrow |C(m,n)| \ll \Delta\, C(m) \left|\log(m/n)\right|^{-1}$$

$$\Rightarrow \left|\sum_{\substack{m\neq n \\ 1\leq m,n\leq T}} \frac{C(m,n)}{\sqrt{mn}}\right| \leq \sum_{m\leq T} \frac{1}{\sqrt{m}} \left(\sum_{\substack{n\leq T \\ n\neq m}} \frac{|C(m,n)|}{\sqrt{n}}\right) \ll \Delta \sum_{m\leq T} \frac{C(m)}{\sqrt{m}} \sum_{\substack{n\leq T \\ n\neq m}} \frac{\left|\log(m/n)\right|^{-1}}{\sqrt{n}}$$

Now, let us prove that:

$$\sum_{\substack{n\le T\\ n\neq m}} \frac{|\log(m/n)|^{-1}}{\sqrt{n}} = \sum_{1\le n<m} \frac{|\log(m/n)|^{-1}}{\sqrt{n}} + \sum_{m<n\le T} \frac{|\log(m/n)|^{-1}}{\sqrt{n}}$$

~~Suppose that~~ Note that when $m=1$, we bound the sum by $O(T^{1/2})$. Then assume $m\ge 2$:

We use integration by parts

$$\sum_{1\le n<m} \frac{|\log(m/n)|^{-1}}{\sqrt{n}} = \sum_{1\le n<m} \frac{1}{\sqrt{n}(\log m/n)} = \sum_{1\le n\le m-1} \frac{1}{\sqrt{n}\log(m/n)}$$

$$= \int_{1^-}^{(m-1)^+} \frac{d[y]}{\sqrt{y}\log(m/y)} = \frac{[m-1]}{\sqrt{m-1}\log(m/m-1)} - \int_1^{m-1} [y]\left(\frac{1}{\sqrt{y}\log(m/y)}\right)' dy = \frac{\sqrt{m-1}}{\log(m/m-1)} - \int_1^{m-1} ([y]-y)\left(\frac{1}{\sqrt{y}\log(m/y)}\right)' dy$$

$$- \int_1^{m-1} y\left(\frac{1}{\sqrt{y}\log(m/y)}\right)' dy$$

$$= \frac{\sqrt{m-1}}{\log(m/m-1)} - \int_1^{m-1} y\left(\frac{1}{\sqrt{y}\log(m/y)}\right)' dy - \int_1^{m-1} ([y]-y)\left(\frac{1}{\sqrt{y}\log(m/y)}\right)' dy$$

(integration by parts)

$$= \frac{\sqrt{m-1}}{\log(m/m-1)} - \left\{\frac{m-1}{\sqrt{m-1}\log(m/m-1)} - \frac{1}{\log m} - \int_1^{m-1} \frac{dy}{\sqrt{y}\log(m/y)}\right\} + O\left(\int_1^{m-1} \left|\left(\frac{1}{\sqrt{y}\log(m/y)}\right)'\right| dy\right)$$

$$= \frac{1}{\log m} + \int_1^{m-1} \frac{dy}{\sqrt{y}\log(m/y)} + O\left(\int_1^{m-1} \left|\left(\frac{1}{\sqrt{y}\log(m/y)}\right)'\right| dy\right) \quad \dots (\alpha)$$

Note that $\displaystyle\int_1^{m-1} \frac{dy}{\sqrt{y}\log(m/y)} \underset{m/y=t}{=} \int_{\frac{m}{m-1}}^{m} \frac{\sqrt{m}\,dt}{t^{3/2}\log t} = \sqrt{m}\int_{1+\frac{1}{m-1}}^{m} \frac{dt}{t^{3/2}\log t} \le \sqrt{m}\int_{1+\frac{1}{m-1}}^{m} \frac{dt}{t\log t}$

$$= \sqrt{m}\,(\log\log t)\Big|_{t=1+\frac{1}{m-1}}^{m} = \sqrt{m}\left(\log\log m - \log\log\left(1+\frac{1}{m-1}\right)\right)$$

$$\Rightarrow \int_1^{m-1} \frac{dy}{\sqrt{y}\log(m/y)} \ll \sqrt{m}\log\log m + \sqrt{m}\cdot\left|\log\log\left(1+\frac{1}{m-1}\right)\right| \quad \dots (*)$$

Since $\frac{C}{m-1} \le \log\left(1+\frac{1}{m-1}\right) \le \frac{1}{m-1}$, for some $C>0 \Rightarrow \log C - \log(m-1) \le \log\log\left(1+\frac{1}{m-1}\right) \le -\log(m$

$$\Rightarrow \text{In } (*): \int_1^{m-1} \frac{dy}{\sqrt{y}\log(m/y)} \le \sqrt{m}\log\log m + \sqrt{m}\left(|\log C| + \log(m-1)\right) \underset{m\le T}{\Downarrow}$$

$$\ll \sqrt{T}\log\log T$$

Clearly $\frac{1}{\log m} \ll 1$. ~~Now we will ... the ...~~

Now we analyze the error term

$$\int_1^{m-1} \left| \left(\frac{1}{\sqrt{y}\,(\log m/y)} \right)' \right| dy \ll \int_1^{m-1} \frac{1}{y^{3/2}(\log m/y)}\,dy + \int_1^{m-1} \frac{dy}{y^{3/2}(\log m/y)^2}$$

Note that $y\log(m/y) \geq C$ $\forall\, y \in [1, m-1]$; for some $C>0$

$$\leq \int_1^{m-1} \frac{dy}{\sqrt{y}\log(m/y)} + C\int_1^{m-1} \frac{dy}{\sqrt{y}\,\log m/y}$$

$$\ll \int_1^{m-1} \frac{dy}{\sqrt{y}\,\log(m/y)} \Leftarrow \text{IT IS THE previous term}$$

So In (α) we get:

$$\sum_{1\leq n<m} \frac{|\log(m/n)|^{-1}}{\sqrt{n}} \ll 1 + \sqrt{T}\log\log T \ll \sqrt{T}\log\log T$$

We analyze the other term:

$$\sum_{m<n\leq T} \frac{|\log(m/n)|^{-1}}{\sqrt{n}} = \sum_{m+1\leq n\leq T} \frac{1}{\sqrt{n}\,(\log n - \log m)}$$

$$= \frac{1}{\sqrt{m+1}\,\log\left(\frac{m+1}{m}\right)} + \sum_{m+2\leq n\leq T} \frac{1}{\sqrt{n}\,(\log n - \log m)}$$

The function is decreasing and we can compare with the integral

$$\leq \frac{1}{\sqrt{m}\,\log(1+\frac{1}{m})} + \int_{m+1}^{T} \frac{dy}{\sqrt{y}\,\log(\underbrace{y/m}_{u})}$$

Using $\log(1+\frac{1}{m}) \geq \frac{C}{m}$, $C>0$

$$\ll \sqrt{m} + \int_{1+\frac{1}{m}}^{T/m} \frac{m\,du}{\sqrt{m}\sqrt{u}\log u} = \sqrt{m} + \sqrt{m}\int_{1+1/m}^{T/m} \frac{du}{\sqrt{u}\log u}$$

$$= \sqrt{m} + \sqrt{m}\int_{1+1/m}^{T/m} \frac{\sqrt{u}\,du}{u\log u} \leq \sqrt{m} + \sqrt{T}\int_{1+1/m}^{T/m} \frac{du}{u\log u}$$

$$\leq \sqrt{m} + \sqrt{T}\left(\log\log u \Big|_{1+1/m}^{T/m} \right)$$

$$\ll \sqrt{m} + \sqrt{T}\Big(\overbrace{\left|\log\log\left(\tfrac{T}{m}\right)\right|}^{\ll \log T} + \underbrace{\left|\log\log\left(1+\tfrac{1}{m}\right)\right|}_{\ll \log m} \Big)$$

$\ll \log T$ ↓ "$m \leq T-1$" (assume)

$\ll \log m \longrightarrow$ as before

$$\ll \sqrt{T}\log T$$

When $T-1 < m \leq T$, we can repeat a similar argument to get the same bound

We conclude that:

$$\sum_{\substack{n \leq T \\ n \neq m}} \frac{|\log(m/n)|^{-1}}{\sqrt{n}} \ll T^{1/2} \log T$$

$\therefore$ We had:

$$\left| \sum_{m \neq n} \frac{C(m,n)}{\sqrt{mn}} \right| \ll \Delta \sum_{m \leq T} \frac{C(m)}{\sqrt{m}} \sum_{\substack{n \leq T \\ n \neq m}} \frac{|\log(m/n)|^{-1}}{\sqrt{n}}$$

$$\ll \Delta \sum_{m \leq T} \frac{C(m)}{\sqrt{m}} T^{1/2} \log T \ll \Delta T^{1/2} \log T \sum_{m \leq T} \frac{C(m)}{\sqrt{m}}$$

Finally we need to bound $\sum_{m \leq T} \frac{C(m)}{\sqrt{m}}$, where $C(m) = \min\left\{\Delta; \left|\log \frac{\pi m^2}{T}\right|^{-1} + \left|\log \frac{2\pi m^2}{T}\right|^{-1}\right\}$

$$\sum_{m \leq T} \frac{C(m)}{\sqrt{m}} = \sum_{m \leq \sqrt{T}} \frac{C(m)}{\sqrt{m}} + \sum_{\sqrt{T} < m \leq T} \frac{C(m)}{\sqrt{m}}$$

$$\leq \underbrace{\sum_{m \leq \sqrt{T}} \frac{\Delta}{\sqrt{m}}}_{} + \sum_{\sqrt{T} < m \leq T} \frac{1}{\sqrt{m}} \left\{ \frac{1}{\left|\log \pi \frac{m^2}{T}\right|} + \frac{1}{\left|\log 2\pi \frac{m^2}{T}\right|} \right\}$$

use integration by parts as before (see p. 4)

$$\ll \Delta \sqrt{T^{1/2}} + T^{1/2} (\log T)^{-1}$$

$$\therefore \left| \sum_{m \neq n} \frac{C(m,n)}{\sqrt{mn}} \right| \ll \Delta T^{1/2} \log T \left(\Delta T^{1/4} + \frac{T^{1/2}}{\log T} \right)$$

$$\ll \underline{\Delta^2 T^{3/4} \log T} + \Delta T \ll \Delta T$$

FIN