Class 2: The Riemann zeta-function in $\mathrm{Re}\, s > 0$

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Review:

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1. For $\mathrm{Re}\, s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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2. Uniformly in compacts of $\mathrm{Re}\, s > 1$, we have that

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$$\sum_{n=1}^{N} \frac{1}{n^s} \to \zeta(s), \quad \text{as} \quad N \to \infty.$$

3. $\zeta(s) \neq 0$ for $\mathrm{Re}\, s > 1$.

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For $\operatorname{Re} s > 1$ we have

$$\begin{aligned}
\zeta(s) - \eta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\
&= \sum_{n=1}^{\infty} \frac{1-(-1)^{n+1}}{n^s} \\
&= \sum_{n:\ \text{even}} \frac{2}{n^s} \\
&= \sum_{k=1}^{\infty} \frac{2}{(2k)^s} \\
&= 2^{1-s} \sum_{k=1}^{\infty} \frac{1}{k^s} = 2^{1-s}\zeta(s)
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4. The function $F(s)$ is an analytic continuation (meromorphic extension) of $\zeta(s)$ in $\mathrm{Re}\, s > 0$.

We will call this $F(s)$ as $\zeta(s)$, because the extension is unique.

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Theorem (Abel's identity)

Let a_n be a sequence of complex numbers., and define the function $A : (0, \infty) \to \mathbb{C}$

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$$\sum_{y < n \le x} a_n f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) \, \mathrm{d}t$$

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$$\sum_{y < n \le x} a_n f(n) = \int_{y^+}^{x^+} f(t)\, \mathrm{d}A(t)$$

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For example

$$\begin{aligned}
\sum_{1\le n\le x} \log n &= \int_{1^-}^{x^+} \log(t)\,\mathrm{d}[t] \\
&= \log x^+[x^+] - \log(1^-)[1^-] - \int_1^x \frac{[t]}{t}\,\mathrm{d}t \\
&= \log x\,[x] - \int_1^x \frac{[t]-t}{t}\,\mathrm{d}t - \int_1^x \frac{t}{t}\,\mathrm{d}t \\
&= x\log x + \log x([x]-x) + \int_1^x \frac{t-[t]}{t}\,\mathrm{d}t - (x-1) \\
&= x\log x - x + 1 + \log x([x]-x) + \int_1^x \frac{t-[t]}{t}\,\mathrm{d}t.
\end{aligned}$$

Let f, g be two functions such that $g(x) \geq 0$ for x large. Then we write

$$f = O(g),$$

if there is $M > 0$ such that $|f(x)| \leq M\, g(x)$ for x large.

$$\sum_{1 \le n \le x} \log n = x \log x - x + 1 + \log x([x] - x) + \int_1^x \frac{t - [t]}{t} \, \mathrm{d}t.$$

$$\sum_{1\le n\le x} \log n = x\log x - x + 1 + \log x([x]-x) + \int_1^x \frac{t-[t]}{t}\,\mathrm{d}t.$$

1. $\left|\log x([x]-x)\right| \le \log x$. Then $\log x([x]-x) = O(\log x)$.

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3. $1 \le \log x$.

$$\sum_{1 \le n \le x} \log n = x \log x - x + 1 + \log x([x] - x) + \int_1^x \frac{t - [t]}{t}\, \mathrm{d}t.$$

1. $\left|\log x([x] - x)\right| \le \log x$. Then $\log x([x] - x) = O(\log x)$.
2. $$\left|\int_1^x \frac{t - [t]}{t}\, \mathrm{d}t\right| \le \int_1^x \frac{1}{t}\, \mathrm{d}t = \log x.$$
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3. $1 \le \log x$. Then $1 = O(\log x)$.

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$$\sum_{1\le n\le x} \log n = x\log x - x + O(\log x).$$

For $\mathrm{Re}\, s > 1$ and $N \geq 2$, using integration by parts:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^s} &= \int_{1^-}^{N^+} \frac{1}{t^s} \mathrm{d}[t] \\
&= \frac{[N^+]}{N^s} - \frac{[1^-]}{1^s} + s \int_1^N \frac{[t]}{t^{s+1}} \mathrm{d}t \\
&= \frac{N}{N^s} + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t + s \int_1^N \frac{t}{t^{s+1}} \mathrm{d}t \\
&= N^{1-s} + s \int_1^N t^{-s} \mathrm{d}t + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t \\
&= N^{1-s} + s \left(\frac{N^{1-s}}{1-s} - \frac{1}{1-s} \right) + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t \\
&= \frac{N^{1-s}}{1-s} + 1 + \frac{1}{s-1} + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t.
\end{aligned}$$

$$\sum_{n=1}^{N} \frac{1}{n^s} = \frac{N^{1-s}}{1-s} + 1 + \frac{1}{s-1} + s \int_1^N \frac{[t]-t}{t^{s+1}} \mathrm{d}t$$

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Then, as $N \to \infty$, we have for $\mathrm{Re}\, s > 1$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t.$$

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The function

$$s \mapsto \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t \text{ is an analytic function in } \mathrm{Re}\, s > 0,$$

Morera's theorem

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Also, in compacts of $\mathrm{Re}\, s > 0$,

$$\left| \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t - \int_1^\infty \frac{[t] - t}{t^{s+1}} \mathrm{d}t \right| \leq \left| \int_N^\infty \frac{1}{t^{s+1}} \mathrm{d}t \right| \leq \int_N^\infty \frac{1}{t^{\sigma_0+1}} \mathrm{d}t \to 0,$$

as $N \to \infty$.

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Therefore, as $N \to \infty$

$$f_N(s) \to \int_1^\infty \frac{[t] - t}{t^{s+1}} \mathrm{d}t,$$

uniformly in compacts of $\mathrm{Re}\, s > 0$.

When $\mathrm{Re}\, s > 1$,

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Therefore, the right-hand side is an analytic continuation of $\zeta(s)$ in $\mathrm{Re}\, s > 0$.

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Then, we have uniformly in compacts of $\mathrm{Re}\, s > 1$:

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right).$$

Taking derivative, we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_p \frac{\log p}{p^s - 1}.$$

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Note that this series converges absolutely for $\mathrm{Re}\, s > 1$.

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For $t \in \mathbb{R}$, we have $\zeta(1+it) \neq 0$.

If $t \neq 0$ it is true. Assume $t \neq 0$. Suppose that $s_0 = 1 + it_0$ ($t_0 \neq 0$) is a zero of order $m \geq 1$ of $\zeta(s)$.

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Demostración.

$$\begin{aligned} 3 + 4\cos\theta + \cos 2\theta &= 3 + 4\cos\theta + 2\cos^2\theta - 1 \\ &= 2\cos^2\theta + 4\cos\theta + 2 = 2(\cos\theta + 1)^2 \geq 0. \end{aligned}$$

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