Class 2: The Riemann zeta-function in $\operatorname{Re} s > 0$

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Review

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1 For $\operatorname{Re} s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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2 Uniformly in compacts of $\operatorname{Re} s > 1$, we have that

$$\sum_{n=1}^{N} \frac{1}{n^s} \to \zeta(s), \text{ as } N \to \infty.$$

3 $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$.

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Analytic continuation of $\zeta(s)$

For $\operatorname{Re} s > 1$ we have

$$\zeta(s) - \eta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$
$$= \sum_{n=1}^{\infty} \frac{1 - (-1)^{n+1}}{n^s}$$
$$= \sum_{n: \text{ even}} \frac{2}{n^s}$$
$$= \sum_{k=1}^{\infty} \frac{2}{(2k)^s}$$
$$= 2^{1-s} \sum_{k=1}^{\infty} \frac{1}{k^s} = 2^{1-s} \zeta(s)$$

Therefore we have for $\operatorname{Re} s > 1$:

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4 The function F(s) is an analytic continuation (meromorphic extension) of $\zeta(s)$ in $\operatorname{Re} s > 0$.

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Analytic continuation of $\zeta(s)$

We will call this F(s) as $\zeta(s)$, because the extension is unique.

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Theorem (Abel's identity)

Let a_n be a sequence of complex numbers., and define the function $A:(0,\infty)\to\mathbb{C}$

$$A(x)=\sum_{n\leq x}a_n,$$

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$$\sum_{y < n \le x} a_n f(n) = \int_{y^+}^{x^+} f(t) \, \mathrm{d}A(t)$$

Using integration by parts:

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$$\sum_{y\leq n\leq x}a_nf(n)=A(x)f(x)-A(y^-)f(y)-\int_y^xf'(t)A(t)\,\mathrm{d}t.$$

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For example

$$\sum_{\leq n \leq x} \log n = \int_{1^{-}}^{x^{+}} \log(t) d[t]$$

= $\log x^{+}[x^{+}] - \log(1^{-})[1^{-}] - \int_{1}^{x} \frac{[t]}{t} dt$
= $\log x [x] - \int_{1}^{x} \frac{[t] - t}{t} dt - \int_{1}^{x} \frac{t}{t} dt$
= $x \log x + \log x([x] - x) + \int_{1}^{x} \frac{t - [t]}{t} dt - (x - 1)$
= $x \log x - x + 1 + \log x([x] - x) + \int_{1}^{x} \frac{t - [t]}{t} dt.$

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Let f, g be two functions such that $g(x) \ge 0$ for x large. Then we write

$$f = O(g),$$

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if there is M > 0 such that $|f(x)| \le M g(x)$ for x large.

$$\sum_{1 \le n \le x} \log n = x \log x - x + 1 + \log x([x] - x) + \int_1^x \frac{t - [t]}{t} dt.$$

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$$\left| \int_1^x \frac{t - [t]}{t} dt \right| \le \int_1^x \frac{1}{t} dt = \log x.$$
Then
$$\left| \int_1^x t - [t] dt \right| \le O(t - x).$$

$$\left|\int_{1}^{x} \frac{t-[t]}{t} \,\mathrm{d}t\right| = O(\log x).$$

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3 $1 \leq \log x$.

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3 $1 \le \log x$. Then $1 = O(\log x)$.

$$\sum_{1 \le n \le x} \log n = x \log x - x + O(\log x).$$

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For Re s > 1 and $N \ge 2$, using integration by parts:

$$\begin{split} \sum_{n=1}^{N} \frac{1}{n^{s}} &= \int_{1^{-}}^{N^{+}} \frac{1}{t^{s}} \mathrm{d}[t] \\ &= \frac{[N^{+}]}{N^{s}} - \frac{[1^{-}]}{1^{s}} + s \int_{1}^{N} \frac{[t]}{t^{s+1}} \mathrm{d}t \\ &= \frac{N}{N^{s}} + s \int_{1}^{N} \frac{[t] - t}{t^{s+1}} \mathrm{d}t + s \int_{1}^{N} \frac{t}{t^{s+1}} \mathrm{d}t \\ &= N^{1-s} + s \int_{1}^{N} t^{-s} \mathrm{d}t + s \int_{1}^{N} \frac{[t] - t}{t^{s+1}} \mathrm{d}t \\ &= N^{1-s} + s \left(\frac{N^{1-s}}{1-s} - \frac{1}{1-s}\right) + s \int_{1}^{N} \frac{[t] - t}{t^{s+1}} \mathrm{d}t \\ &= \frac{N^{1-s}}{1-s} + 1 + \frac{1}{s-1} + s \int_{1}^{N} \frac{[t] - t}{t^{s+1}} \mathrm{d}t. \end{split}$$

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$$\sum_{n=1}^{N} \frac{1}{n^{s}} = \frac{N^{1-s}}{1-s} + 1 + \frac{1}{s-1} + s \int_{1}^{N} \frac{[t]-t}{t^{s+1}} \mathrm{d}t$$

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Then, as $N o \infty$, we have for $\operatorname{Re} s > 1$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t.$$

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The function

 $s\mapsto \int_1^\infty rac{[t]-t}{t^{s+1}}\mathrm{d}t$ is an analytic function in $\operatorname{Re} s>0,$

Morera's theorem

The function f_N is analytic in Re s > 0:

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Also, in compacts of $\operatorname{Re} s > 0$,

$$\left|\int_{1}^{N} \frac{[t]-t}{t^{s+1}} \mathrm{d}t - \int_{1}^{\infty} \frac{[t]-t}{t^{s+1}} \mathrm{d}t\right| \leq \left|\int_{N}^{\infty} \frac{1}{t^{s+1}} \mathrm{d}t\right| \leq \int_{N}^{\infty} \frac{1}{t^{\sigma_{0}+1}} \mathrm{d}t \to 0,$$

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$$f_N(s) \rightarrow \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t,$$

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The function f_N is analytic in Re s > 0:

$$f_N(s) = \int_1^N \frac{[t]-t}{t^{s+1}} \mathrm{d}t.$$

Also, in compacts of $\operatorname{Re} s > 0$,

$$\left| \int_{1}^{N} \frac{[t] - t}{t^{s+1}} \mathrm{d}t - \int_{1}^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t \right| \le \left| \int_{N}^{\infty} \frac{1}{t^{s+1}} \mathrm{d}t \right| \le \int_{N}^{\infty} \frac{1}{t^{\sigma_{0}+1}} \mathrm{d}t \to 0,$$

as $N \to \infty$.

Therefore, as $N
ightarrow \infty$

$$f_N(s) \rightarrow \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t,$$

uniformly in compacts of $\operatorname{Re} s > 0$.

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When $\operatorname{Re} s > 1$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t$$

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When $\operatorname{Re} s > 1$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t.$$

Therefore, the right-hand side is an analytic continuation of $\zeta(s)$ in $\operatorname{Re} s > 0$.

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Class 2: The Riemann zeta-function in Re s > 0

L The Riemann zeta-function in $\operatorname{Re} s = 1$

We have uniformly in compacts of $\operatorname{Re} s > 1$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

Class 2: The Riemann zeta-function in Re s > 0

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$$\zeta(s) = \prod_p \left(1 - rac{1}{p^s}
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Then, we have uniformly in compacts of $\operatorname{Re} s > 1$:

$$\log \zeta(s) = -\sum_p \log \left(1 - \frac{1}{p^s}\right).$$

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Taking derivative, we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p} \frac{\log p}{p^s - 1}.$$

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Note that for $\operatorname{Re} s > 1$:

$$\sum_p \left| rac{\log p}{p^s - 1}
ight| \leq \sum_p rac{\log p}{p^\sigma - 1} = -rac{\zeta'}{\zeta}(\sigma) < \infty.$$

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Then, we can reorder the series:

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p} \frac{\log p}{p^s - 1}.$$

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p} \frac{\log p}{p^s - 1}$$

Note that for $\operatorname{Re} s > 1$:

$$\frac{1}{p^{s}-1} = \frac{p^{-s}}{1-p^{-s}} = p^{-s} \sum_{k=0}^{\infty} (p^{-s})^{k} = \sum_{k=1}^{\infty} (p^{-s})^{k}.$$

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Class 2: The Riemann zeta-function in Re s > 0

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Therefore

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p} \sum_{k=1}^{\infty} \frac{\log p}{p^{sk}}.$$

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Class 2: The Riemann zeta-function in Re s > 0

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Therefore, we can write, for $\operatorname{Re} s > 1$:

$$-rac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} rac{\Lambda(n)}{n^s},$$

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Class 2: The Riemann zeta-function in Re s > 0

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$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{if } n \neq p^k. \end{cases}$$

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Note that this series converges absolutely for $\operatorname{Re} s > 1$.

Theorem (Hadamard, de la Vallée -Poussin 1896)

For $t \in \mathbb{R}$, we have $\zeta(1 + it) \neq 0$.

If $t \neq 0$ it is true. Assume $t \neq 0$. Suppose that $s_0 = 1 + it_0$ $(t_0 \neq 0)$ is a zero of order $m \geq 1$ of $\zeta(s)$.

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$$rac{\zeta'}{\zeta}(\sigma+it_0)=rac{m}{\sigma-1}+rac{\mathcal{A}'(\sigma+it_0)}{\mathcal{A}(\sigma+it_0)} \ \ \, ext{around} \ \ \, \sigma>1.$$

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$\zeta(s)$ has a unique simple pole in s = 1. Then,

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

$\zeta(s)$ is analytic in $s_2 = 1 + 2it_0$.

L The Riemann zeta-function in $\operatorname{Re} s = 1$

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 is analytic in $s_2 = 1 + 2it_0$.
1 If $s = 1 + 2it_0$ is a zero (of order $k \ge 1$)

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

$$\begin{split} \zeta(s) \text{ is analytic in } s_2 &= 1 + 2it_0. \\ & \blacksquare \text{ If } s = 1 + 2it_0 \text{ is a zero (of order } k \geq 1) \\ & \zeta(s) &= (s - s_2)^k C(s), \quad C(s_2) \neq 0; \end{split}$$

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$$\frac{\zeta'}{\zeta}(s) = \frac{k}{s-s_2} + \frac{C'(s)}{C(s)}$$
 around s_2 .

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

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If $s = 1 + 2it_0$ is a zero (of order $k \ge 1$)
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 $\frac{\zeta'}{\zeta}(\sigma + 2it_0) = \frac{k}{\sigma - 1} + \frac{C'(\sigma + 2it_0)}{C(\sigma + 2it_0)} \text{ around } \sigma > 1.$

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$$\frac{\zeta'}{\zeta}(\sigma + 2it_0) &= \frac{k}{\sigma - 1} + \frac{C'(\sigma + 2it_0)}{C(\sigma + 2it_0)} \text{ around } \sigma > 1. \end{aligned}$$

2 If $s = 1 + 2it_0$ is not a zero, consider k = 0 and $C = \zeta$.

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

Resumming: around $\sigma > 1$

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Resumming: around $\sigma > 1$

1 With $m \ge 1$ we have

$$\frac{\zeta'}{\zeta}(\sigma+it_0) = \frac{m}{\sigma-1} + \frac{A'(\sigma+it_0)}{A(\sigma+it_0)}; \qquad \frac{\zeta'}{\zeta}(\sigma) = \frac{B'(\sigma)}{B(\sigma)} - \frac{1}{\sigma-1}.$$

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Re $\left\{3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma+it_0) + \frac{\zeta'}{\zeta}(\sigma+2it_0)\right\} = \frac{-3+4m+k}{\sigma-1}$ +bounded.

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Resumming: around $\sigma > 1$

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Then, when $\sigma \rightarrow 1^+$, we have that

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Resumming: around $\sigma > 1$

1 With $m \ge 1$ we have

$$\frac{\zeta'}{\zeta}(\sigma+it_0) = \frac{m}{\sigma-1} + \frac{A'(\sigma+it_0)}{A(\sigma+it_0)}; \qquad \frac{\zeta'}{\zeta}(\sigma) = \frac{B'(\sigma)}{B(\sigma)} - \frac{1}{\sigma-1}.$$

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Then, when $\sigma \rightarrow 1^+$, we have that

$$\operatorname{Re}\left\{3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma + it_0) + \frac{\zeta'}{\zeta}(\sigma + 2it_0)\right\} > 0$$

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Lemma

For any $\theta \in \mathbb{R}$ we have: $3 + 4\cos\theta + \cos 2\theta \ge 0$.

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Lemma

For any
$$\theta \in \mathbb{R}$$
 we have: $3 + 4\cos\theta + \cos 2\theta \ge 0$.

Demostración.

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1$$

= $2\cos^2\theta + 4\cos\theta + 2 = 2(\cos\theta + 1)^2 \ge 0.$

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

Recall that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Recall that

$$-\frac{\zeta'}{\zeta}(s)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.$$

Then

$$\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(s)\right\} = -\operatorname{Re}\left\{\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma+it}}\right\}$$

The Riemann zeta-function in $\operatorname{Re} s = 1$

Recall that $-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$ Then $\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(s)\right\} = -\operatorname{Re}\left\{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma+it}}\right\}$ $\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(s)\right\} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \operatorname{Re}\left\{\frac{1}{n^{it}}\right\} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \cos(t \log n).$

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L The Riemann zeta-function in $\operatorname{Re} s = 1$

Since, for $\sigma > 1$:

$$\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(s)\right\} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \cos(t \log n),$$

L The Riemann zeta-function in $\operatorname{Re} s = 1$

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$$\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(s)\right\} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \cos(t \log n),$$

one can see that

$$\operatorname{Re}\left\{3\frac{\zeta'}{\zeta}(\sigma)+4\frac{\zeta'}{\zeta}(\sigma+it_0)+\frac{\zeta'}{\zeta}(\sigma+2it_0)\right\}$$

Class 2: The Riemann zeta-function in Re s > 0

L The Riemann zeta-function in $\operatorname{Re} s = 1$

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$$= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \left(3 + 4\cos(t_0\log n) + \cos(2t_0\log n)\right)$$

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$$= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \left(3 + 4\cos(t_0\log n) + \cos(2t_0\log n)\right)$$
$$< 0.$$

L The Riemann zeta-function in $\operatorname{Re} s = 1$

Since, for $\sigma > 1$:

$$\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(s)\right\} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \cos(t \log n),$$

one can see that

$$\operatorname{Re}\left\{3\frac{\zeta'}{\zeta}(\sigma) + 4\frac{\zeta'}{\zeta}(\sigma + it_0) + \frac{\zeta'}{\zeta}(\sigma + 2it_0)\right\}$$
$$= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \left(3 + 4\cos(t_0\log n) + \cos(2t_0\log n)\right)$$

 \leq 0.

