Class 32: The Riemann zeta function 1859

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

$x>1$: $\displaystyle\sum_{n=1}^{N} \frac{1}{n^x}$

$n-1 \le t \le n$, $t \in [n-1, n]$

$\frac{1}{n} \le \frac{1}{t} \Rightarrow \frac{1}{n^x} \le \frac{1}{t^x} \Rightarrow \int_{n-1}^{n} \frac{dt}{n^x} \le \int_{n-1}^{n} \frac{1}{t^x}\,dt$

$\Rightarrow \frac{1}{n^x} \le \int_{n-1}^{n} \frac{1}{t^x}\,dt$

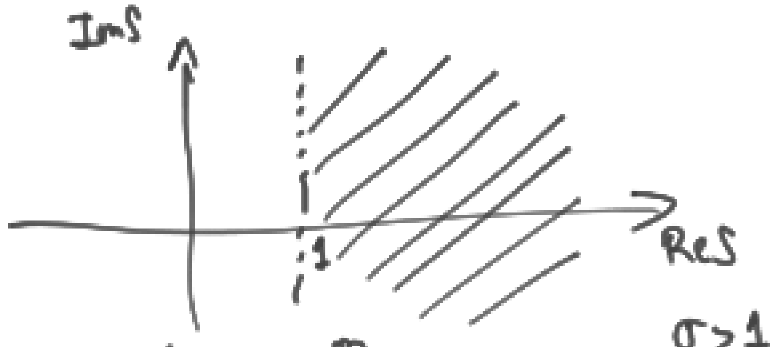
$$\sum_{n=2}^{N} \frac{1}{n^x} \le \sum_{n=2}^{N} \int_{n-1}^{n} \frac{1}{t^x}\,dt = \int_1^N \frac{1}{t^x}\,dt = \left.\frac{t^{-x+1}}{-x+1}\right|_{t=1}^{t=N} = \boxed{\frac{N^{-x+1}}{-x+1}} + \frac{1}{x-1}$$

$-x+1<0$, $N \to \infty$

$F(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$ is convergent for $x>1$

$x \rightsquigarrow s$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$



$|n^s| = n^{\operatorname{Re} s}$

$s = \sigma + it$

$\sigma > 1$

Re s > 1: $\displaystyle\sum_{n=1}^{\infty} \left\| \frac{1}{n^s} \right\| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ is finite

$\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$; defined in $\operatorname{Re} s > 1$

ζ is ABSOLUTELY CONVERGENT in $\operatorname{Re} s > 1$.

$\zeta_N(s) = \displaystyle\sum_{n=1}^{N} \frac{1}{n^s}$; analytic in $\operatorname{Re} s > 1$

$\zeta_N(s) \longrightarrow \zeta(s)$; when $N \to \infty$, for any fixed $s \in \mathbb{C}$, $\operatorname{Re} s > 1$.

Let $\varepsilon > 0$; consider $\mathcal{H}_\varepsilon = \{ s \in \mathbb{C} : \operatorname{Re} s \geq 1 + \varepsilon \}$

(Fixed)

$\zeta_N \longrightarrow \zeta$ uniformly in $\mathcal{H}$ $\qquad \sigma \geq 1+\varepsilon$

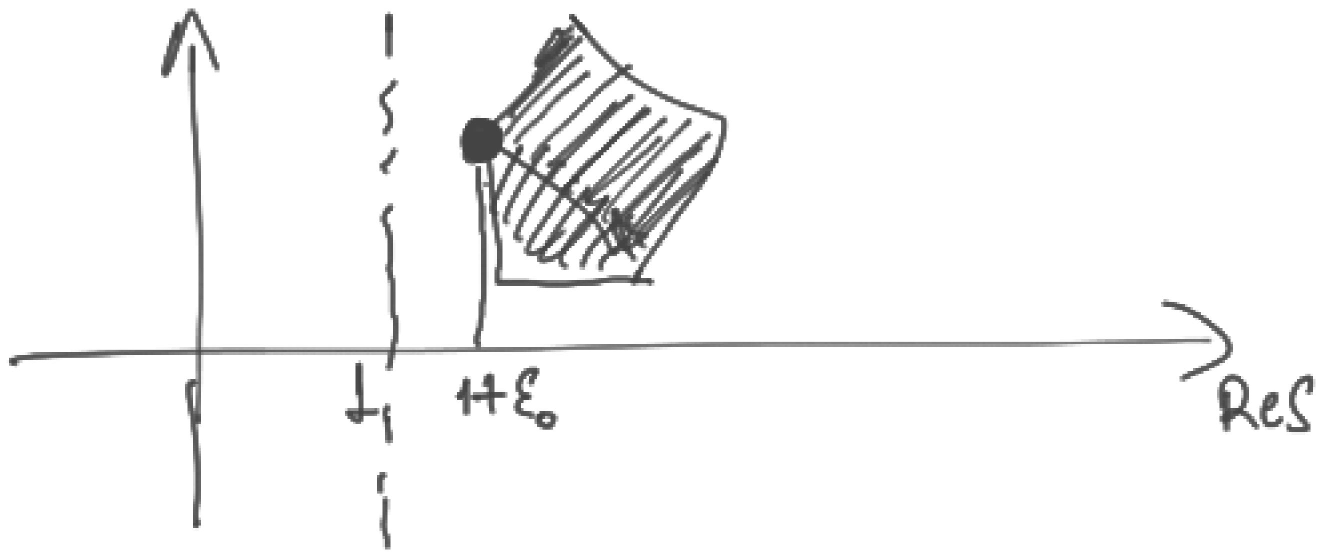
$\left| \frac{1}{n^s} \right| = \frac{1}{n^\sigma} \le \frac{1}{n^{1+\varepsilon}}$ $\qquad \displaystyle\sum_{n\geq 1} \frac{1}{n^{1+\varepsilon}}$ is convergent

$\Rightarrow$ T. Weierstrass $\displaystyle\sum_{n=1}^{N} \frac{1}{n^s}$ is uniformly convergent in $\mathcal{H}$

$\zeta_N(s) \xrightarrow[\mathcal{H}]{\text{uniformly}} \zeta(s)$

$\zeta_N(s)$ is uniformly conv. in $\mathcal{H}$

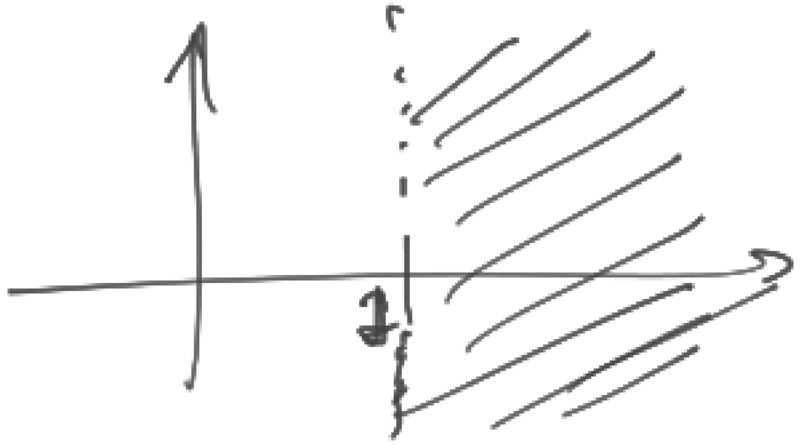
Let K a compact set in $\mathrm{Re}\, s > 1$



$$K \subset \{ s \in \mathbb{C} : \mathrm{Re}\, s \geq 1 + \varepsilon_0 \}$$

$$\zeta_N(s) \xrightarrow[\text{(in } K)]{\text{uniformly}} \zeta(s) \text{ in } H_{\varepsilon_0}$$ (in particular uniformly in K)

ζ is an analytic function in $\mathrm{Re}\, s > 1$.



$$\lim_{\sigma \to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} = \infty$$

$$\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

PROOF: $\tilde{\zeta}_N(s) \to \tilde{\zeta}(s)$ uniformly in compacts of $\mathrm{Re}\, s > 0$

$\tilde{\zeta}$ is an analytic function in $\mathrm{Re}\, s > 0$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \qquad \sigma > 1$$

$$\left. \sum_{n=1}^{\text{"}\infty\text{"}} \frac{1}{n^{\sigma}} \ ; \ \sigma \leq 1 \right\}$$ is No CONVERGENT

$$\sum_{n=1}^{\text{"}\infty\text{"}} \frac{1}{\sqrt{n}}$$ is NOT CONVERGENT

$\zeta_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}$; $\tilde{\zeta}_N(s) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}$ $\quad \zeta_N(s) \to \zeta(s)$, $\mathrm{Re}\, s > 1$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$; $\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ $\quad \tilde{\zeta}_N(s) \to \tilde{\zeta}(s)$

$\mathrm{Re}\, s > 1$

$$\zeta_{2N}(s) - \tilde{\zeta}_{2N}(s) = \sum_{n=1}^{2N} \frac{1-(-1)^{n+1}}{n^s} = \sum_{\substack{k=1 \\ \boxed{n=2k-1}}}^{N} \frac{1-(-1)^{2k-1+1}}{(2k-1)^s} \Big\} \text{"}0\text{"}$$

$$+ \sum_{\substack{k=1 \\ \boxed{n=2k}}}^{N} \frac{1-(-1)^{2k+1}}{(2k)^s} \quad 1-(-1) \checkmark$$

$$= \sum_{k=1}^{N} \frac{2}{(2k)^s}$$

$$\zeta_{2N}(s) - \tilde{\zeta}_{2N}(s) = \frac{2}{2^s} \sum_{k=1}^{N} \frac{1}{k^s} = 2^{1-s} \cdot \zeta_N(s)$$

$$\zeta_{2N}(s) - \tilde{\zeta}_{2N}(s) = 2^{1-s} \zeta_N(s), \text{ for } \mathrm{Re}\, s > 1$$

$\downarrow N \to \infty$

$$\zeta(s) - \tilde{\zeta}(s) = 2^{1-s} \cdot \zeta(s)$$

$$\zeta(s) - 2^{1-s} \zeta(s) = \tilde{\zeta}(s)$$

$$\boxed{\zeta(s)(1-2^{1-s}) = \tilde{\zeta}(s)} \; ; \mathrm{Re}\, s > 1$$

$$\boxed{\zeta(s) = \frac{\tilde{\zeta}(s)}{1-2^{1-s}}} \quad \mathrm{Re}\, s > 1$$

EXTENDS ζ FOR $\mathrm{Re}\, s > 0$

$$\boxed{1 - 2^{1-s} = 0} \Rightarrow 1 = 2^{1-s}$$

$s = 1$

POLE

$\lambda > 0$:

- $G_\lambda(t) = e^{-\pi\lambda t^2} \Rightarrow \widehat{G}_\lambda(x) = \lambda^{-1/2} e^{-\pi \frac{x^2}{\lambda}} \leftarrow$

PSM: $\sum_{n\in\mathbb{Z}} G_\lambda(x+n) = \sum_{k\in\mathbb{Z}} \widehat{G}_\lambda(k) e^{2\pi i k x}$ $\quad x=0$

$$\boxed{\sum_{n\in\mathbb{Z}} G_\lambda(n) = \sum_{k\in\mathbb{Z}} \widehat{G}_\lambda(k)}$$

$$\sum_{n\in\mathbb{Z}} G_\lambda(n) = \sum_{n\in\mathbb{Z}} e^{-\pi\lambda n^2} = 1 + 2\underbrace{\sum_{n\geq 1} e^{-\pi\lambda n^2}} = 1 + 2\theta(\lambda)$$

$$\sum_{k\in\mathbb{Z}} \widehat{G}_\lambda(k) = \lambda^{-1/2}\Big(1 + 2\underbrace{\sum_{k\geq 1} e^{-\pi\frac{k^2}{\lambda}}}\Big)$$

$$= \lambda^{-1/2}\left(1 + 2.\theta\left(\tfrac{1}{\lambda}\right)\right)$$

$$\underbrace{1 + 2\theta(\lambda) = \lambda^{-1/2}\left(1 + 2\theta\left(\tfrac{1}{\lambda}\right)\right)}; \quad \theta(\lambda) = \sum_{n\geq 1} e^{-\pi\lambda n^2}$$

$\lambda > 1$: $0 < \theta(\lambda) = \sum_{n\geq 1} e^{-\pi\lambda n^2} \leq \sum_{n\geq 1} e^{-\pi\lambda n} = \dfrac{e^{-\pi\lambda}}{1 - e^{-\pi\lambda}}$

$\lambda > 1 \Rightarrow -\lambda < -1 \Rightarrow -\pi\lambda < -\pi$

$e^{-\pi\lambda} < e^{-\pi}$

$-e^{-\pi\lambda} > -e^{-\pi} \Rightarrow 1 - e^{-\pi\lambda} > 1 - e^{-\pi}$

$\dfrac{1}{1-e^{-\pi\lambda}} < \dfrac{1}{1-e^{-\pi}} \Leftarrow$

$\lambda > 1$: $\boxed{0 < \theta(\lambda) \leq \dfrac{e^{-\pi\lambda}}{1-e^{-\pi}}}$

$\boxed{\bullet\ \lambda > 1:\ 0 < \theta(\lambda) \leq C e^{-\pi\lambda}, \text{ for some constant}}$

When $\lambda < 1$; so $\frac{1}{\lambda} > 1$

$\lambda < 1$: $1 + 2\theta(\lambda) \leq \lambda^{-1/2}\left(1 + 2.\dfrac{e^{-\pi/\lambda}}{1-e^{-\pi}}\right) \leq \lambda^{-1/2}\left(1 + \dfrac{2}{1-e^{-\pi}}\right) = \lambda^{-1/2}\left(\dfrac{3 - e^{-\pi}}{1-e^{-\pi}}\right)$

$\lambda \leq 1$:

$$1 + 2\theta(\lambda) \leq \lambda^{-1/2} \left(\frac{3 - e^{-\pi}}{1 - e^{-\pi}} \right)$$

$2\theta(\lambda) \leq$

$$2\theta(\lambda) \leq \lambda^{-1/2} \cdot C$$

$$\theta(\lambda) \leq \lambda^{-1/2} \cdot \frac{C}{2}$$

$$0 < \theta(\lambda) \leq D \cdot \lambda^{-1/2} \; ; \; \lambda \leq 1$$

Condussion:

$\lambda > 1$: $\quad 0 < \theta(\lambda) \leq C \cdot e^{-\pi\lambda}$

$\lambda \leq 1$; $\quad 0 < \theta(\lambda) \leq D \lambda^{-1/2}$

for some $C, D > 0$ $\longrightarrow$

Re $s > 0$:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \qquad \text{Fix } n \geq 1$$

$t = \pi \lambda n^2$

$$\Gamma(s) = \int_0^\infty e^{-\pi\lambda n^2} \cdot (\pi \lambda n^2)^{s-1} \cdot \pi n^2 d\lambda$$

$$= \int_0^\infty e^{-\pi\lambda n^2} \cdot \pi^s \cdot \lambda^{s-1} \cdot n^{2(s-1)+2} d\lambda$$

$$\Gamma(s) = \int_0^\infty e^{-\pi\lambda n^2} \cdot \pi^s \cdot \lambda^{s-1} \cdot n^{2s} d\lambda$$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\pi\lambda n^2} \cdot \pi^{\frac{s}{2}} \lambda^{\frac{s}{2}-1} \cdot n^s d\lambda$$

Re $s > 0$: $\pi^{-s/2} \cdot \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty e^{-\pi\lambda n^2} \cdot \lambda^{\frac{s}{2}-1} d\lambda$, $n \geq 1$

In particular for Re $s > 1$: Summing for $1 \leq n$:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n \geq 1} \int_0^\infty e^{-\pi\lambda n^2} \cdot \lambda^{s/2 - 1} d\lambda$$

Class 35: Riemann zeta-function and Uncertainty principle

Remember:

*) $\Gamma(s)$ is a meromorphic function with simple poles at $s = 0, -1, -2, -3, \dots$

$\Gamma(0+1) = \Gamma(1) = 1$

$\Gamma(s+1) = s\Gamma(s) \Rightarrow \frac{\Gamma(s+1)}{s} = \Gamma(s)$

$\mathrm{Re}\, s > 0$ $\qquad$ $\mathrm{Re}\, s > -1$

SIMPLE POLE

Γ has zeros?

*) $\frac{1}{\Gamma(s)}$ is an entire function with simple zeros at $s = 0, -1, -2, -3, \dots$

$s = 0, -1, -2$

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}} \Rightarrow \frac{\Gamma(s) \sin \pi s}{\pi} = \frac{1}{\Gamma(s)}$$

$0 < \mathrm{Re}\, s < 1$ $\qquad$ $\sin \pi s$

Define:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \text{analytic in } \mathrm{Re}\, s > 1.$$

DIRICHLET ETA-FUNCTION $\rightarrow$ $$\tilde{\zeta}(s) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s} \quad \text{analytic in } \mathrm{Re}\, s > 0$$

$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^s}$ uniformly in compacts

We have proved:

$$\zeta(s) = \frac{\tilde{\zeta}(s)}{1 - 2^{1-s}}; \quad \mathrm{Re}\, s > 1$$

$\zeta(s)$ $\qquad$ $\tilde{\zeta}$

$\frac{(-1)^{n+1}}{n^s} + \frac{(-1)^{n+2}}{(n+1)^s}$

$\frac{1}{n^s} - \frac{1}{(n+1)^s} = \frac{(n+1)^s - n^s}{n^s (n+1)^s} \approx \frac{n^{s-1}}{n^s (n+1)^s} \approx \frac{1}{n^{s+1}}$ $\quad$ $\mathrm{Re}\, s > 0$

$1 = 2^{s-1}$

$1 = e^{(s-1)\ln 2}$

$(s-1)\ln 2 = 2\pi k, \quad k \in \mathbb{Z}$

POLES: $\boxed{1 - 2^{1-s} = 0}$ $\qquad$ $\boxed{s = \frac{2\pi k}{\ln 2} + 1}$?, $k \in \mathbb{Z}$, $k \neq 0$

$k = 0;\ s = 1;\quad \tilde{\zeta}(1) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$

$$\Theta(\lambda) = \sum_{n\geq 1} e^{-\pi\lambda n^2}$$

$C, D \in \mathbb{R}$

- $\boxed{0 < \Theta(\lambda) \leq C e^{-\pi\lambda}}$, for $\lambda > 1$
- $0 < \Theta(\lambda) \leq D\lambda^{-1/2}$, for $0 < \lambda \leq 1$
- $1 + 2\Theta(\lambda) = \lambda^{-1/2}\left(1 + 2\Theta\left(\frac{1}{\lambda}\right)\right)$ for $\lambda > 0$ (Poisson Summation F.)

MODULAR EQUATION

$\mathrm{Re}\, s > 0$: $\boxed{\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt}$ $t = \pi\lambda n^2$, $n \geq 1$, "$\lambda > 0$"

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\pi\lambda n^2} \pi^{\frac{s}{2}} \lambda^{\frac{s}{2}-1} n^s\, d\lambda$$

WE HAVE PROVED

For $\mathrm{Re}\, s > 1$:

$$\pi^{-s/2}\Gamma(s/2)\, n^{-s} = \int_0^\infty e^{-\pi\lambda n^2} \lambda^{\frac{s}{2}-1}\, d\lambda$$

SUMMING OVER ALL $n \geq 1$

$\sum_{n\geq 1}$ $\qquad$ $\sum_{n\geq 1}$

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n\geq 1}\int_0^\infty e^{-\pi\lambda n^2}\lambda^{\frac{s}{2}-1}\, d\lambda$$

CHANGE

$$= \int_0^\infty \sum_{n\geq 1} e^{-\pi\lambda n^2} \lambda^{\frac{s}{2}-1}\, d\lambda$$

$$\lim_{N\to\infty}\sum_{n\geq 1}^{N}\int_0^\infty e^{-\pi\lambda n^2}\lambda^{\frac{s}{2}-1}\, d\lambda = \int_0^\infty \lim_{N\to\infty}\sum_{n\geq 1}^{N} e^{-\pi\lambda n^2}\lambda^{\frac{s}{2}-1}\, d\lambda$$

EXERCISE

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda \quad ; \quad \mathrm{Re}\, s > 1$$

$$= \int_0^1 \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda$$

$$= \int_0^1 \frac{1}{2}\left[\lambda^{-\frac{1}{2}}\left(1 + 2\Theta\left(\frac{1}{\lambda}\right)\right) - 1\right]\lambda^{\frac{s}{2}-1}\, d\lambda + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda$$

$\lambda^{-1} = \frac{1}{\lambda} = t$, $\lambda = t^{-1}$

$$= \int_1^\infty \frac{1}{2}\left[t^{1/2}\left(1 + 2\Theta(t)\right) - 1\right] \cdot t^{1-\frac{s}{2}}\frac{dt}{t^2} + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda$$

$$= \frac{1}{2}\int_1^\infty \left(t^{1/2} + 2t^{1/2}\Theta(t) - 1\right) t^{-1-\frac{s}{2}}\, dt + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda$$

$$= \frac{1}{2}\left[\int_1^\infty t^{-\frac{1}{2}-\frac{s}{2}}\, dt + \int_1^\infty 2\Theta(t)t^{-\frac{1}{2}-\frac{s}{2}}\, dt - \int_1^\infty t^{-1-\frac{s}{2}}\, dt\right] + \ldots$$

$$= \frac{1}{2}\left[\frac{t^{\frac{1}{2}-\frac{s}{2}}}{\frac{1}{2}-\frac{s}{2}}\Bigg|_1^\infty - \frac{t^{-s/2}}{-s/2}\Bigg|_1^\infty\right] + \int_1^\infty \Theta(\lambda)\lambda^{-\frac{1}{2}-\frac{s}{2}}\, d\lambda + \int_1^\infty \Theta(\lambda)\lambda^{\frac{s}{2}-1}\, d\lambda$$

$$= \frac{1}{2}\left[-\frac{1}{\frac{1}{2}-\frac{s}{2}} - \left\{-\frac{1}{-s/2}\right\}\right] = \frac{1}{s-1} - \frac{1}{s} + \ldots + \ldots$$

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \underbrace{\int_1^\infty \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda}_{g(s)}$$

$\mathrm{Re}\, s>1$

$$g(s) = \int_1^\infty \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda$$

$$g_n(s) = \int_1^n \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda$$

$g_n(s) - g_n(s_0) \to 0$

•) g_n is continuous

• $$\int_{\partial\Delta} g_n(s)ds = \int_{\partial\Delta}\left(\int_1^n \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda\right)ds$$

$$= \int_1^n \theta(\lambda)\int_{\partial\Delta}\underbrace{\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)ds}_{0}\, d\lambda = 0$$

$\Rightarrow$ MORERA TH. $\Rightarrow$ g_n is analytic in $\mathbb{C}$

g_n is entire

g is an entire function (analytic on $\mathbb{C}$)

$$\zeta(s) = \left(\underset{\mathbb{C}}{\frac{1}{s-1}} - \underset{\mathbb{C}}{\frac{1}{s}} + \underset{\mathbb{C}}{g(s)}\right)\underset{\mathbb{C}}{\pi^{s/2}} \cdot \underset{\mathbb{C}}{\frac{1}{\Gamma(\frac{s}{2})}}, \quad \mathrm{Re}\, s>1$$

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}, \quad \mathrm{Re}\, s>1$$

$\frac{1}{s}$ $\frac{1}{\Gamma(\frac{s}{2})}$ has a zero at $s=0$

$\frac{1}{s\Gamma(s/2)}$

$\dfrac{1/\Gamma(s/2)}{s}$ doesn't have zero at $s=0$

•) ζ is meromorphic function in $\mathbb{C}$.

•) ζ has a simple pole at $\underline{\underline{s=1}}$.

•) ζ has zeros at $s=-2,-4,-6,\dots$ (zeros of $1/\Gamma(s/2)$).

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \boxed{\frac{1}{s-1} - \frac{1}{s} + g(s)}$$

$\frac{1}{(1-s)-1} - \frac{1}{1-s} + g(1-s)$

$-\frac{1}{s} + \frac{1}{s-1} + g(s)$ $\mathrm{Re}\, s > 1$

where $g(s) = \int_1^\infty \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{s}{2}} + \lambda^{\frac{s}{2}-1}\right)d\lambda$ entire function

- ζ has a meromorphic continuation in $\mathbb{C}$.
- ζ has only a simple pole at $s=1$.

$\left|\frac{1}{p^s}\right| = \frac{1}{p^{\mathrm{Re}\, s}} \leq \frac{1}{p}$

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \dots \frac{1}{n^s} + \dots = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \dots\right)$

$\mathrm{Re}\, s > 1$

$1 + x + x^2 + \dots$

$= \prod_p \frac{1}{1-\frac{1}{p^s}} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$

uniform in compacts

$\zeta(s)$ has no zeros in $\mathrm{Re}\, s > 1$

$\left|\frac{1}{p^s}\right| < 1$

$$g(1-s) = \int_1^\infty \theta(\lambda)\left(\lambda^{-\frac{1}{2}-\frac{(1-s)}{2}} + \lambda^{\frac{1-s}{2}-1}\right)d\lambda = \underbrace{\int_1^\infty \theta(\lambda)\left(\lambda^{-1+\frac{s}{2}} + \lambda^{-\frac{1}{2}-\frac{s}{2}}\right)d\lambda}_{= g(s)}$$

$g(1-s) = g(s)$ for $s \in \mathbb{C}$

$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + g(s)$ holds for $s \in \mathbb{C}$

$$\boxed{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}$$

for all $s \in \mathbb{C}$

$A = B$

FUNCTIONAL EQUATION

- $\zeta(s)$ has no zeros in $\mathrm{Re}\, s > 1$
- In $\mathrm{Re}\, s < 0$:

$\mathrm{Re}(1-s) > 1$

$\mathrm{Re}\left(\frac{1-s}{2}\right) > \frac{1}{2}$

$\mathrm{Re}\, s < 0$

$\Gamma\left(\frac{1-s}{2}\right)$ is analytic in $\mathrm{Re}\, s < 0$

$\zeta(1-s)$ is analytic in $\mathrm{Re}\, s < 0$

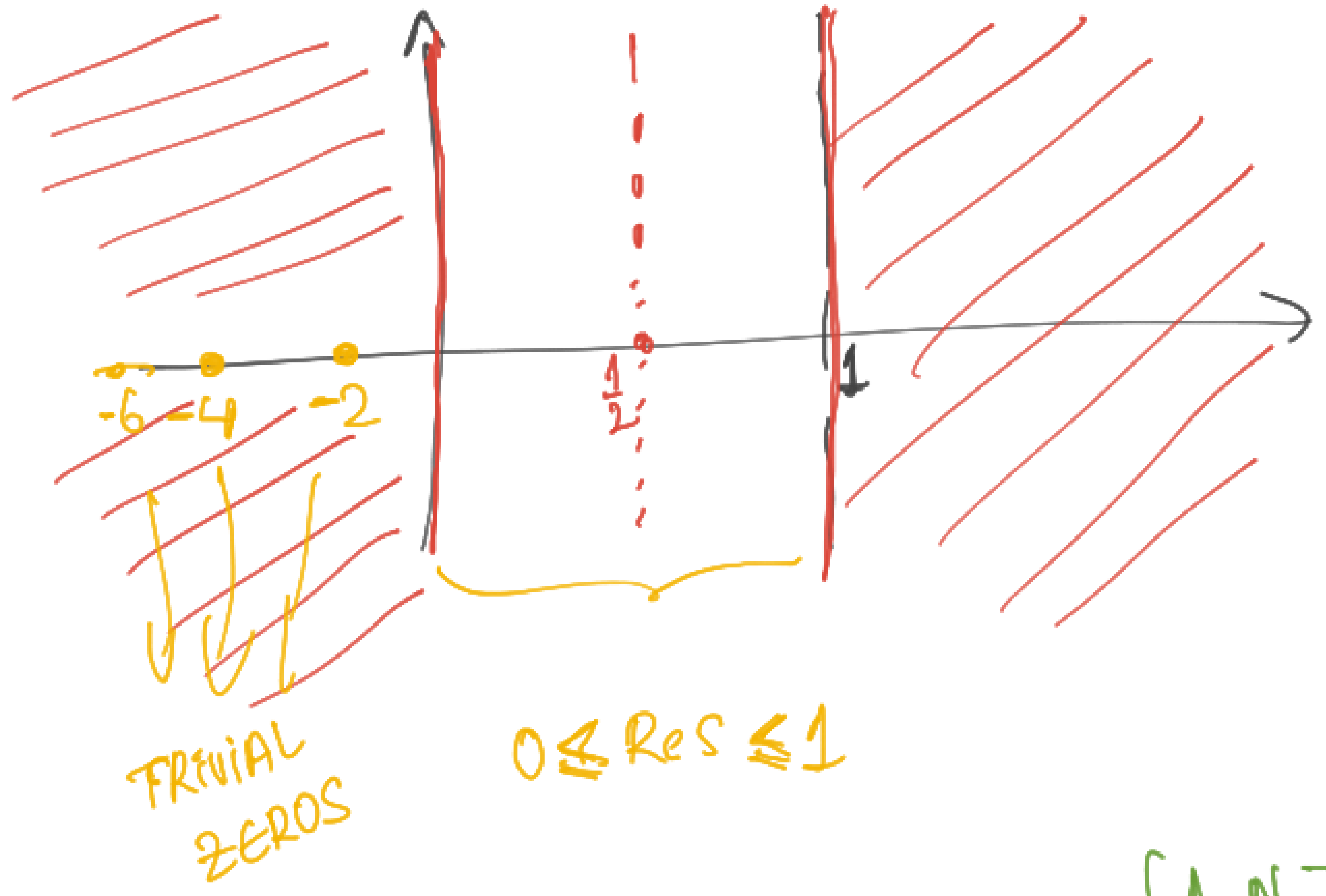
$\Rightarrow$ B is analytic in $\mathrm{Re}\, s < 0$

$\Rightarrow$ A is analytic in $\mathrm{Re}\, s < 0$

$\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is analytic in $\mathrm{Re}\, s < 0$ $\longrightarrow$ ζ has zeros in $s = -2, -4, -6, \dots$

poles in $s = -2, -4, -6, \dots$

SIMPLE ZEROS



- $\mathrm{Re}\,S = 1$, ζ has no zeros (A.N.T.)
- $\mathrm{Re}\,S = 0$, ζ has no zeros (functional equation)

$\boxed{16:15\text{ PM}} - \boxed{17:00}$

$0 < \mathrm{Re}\,S < 1$ → CRITICAL STRIP

1859

RIEMANN HYPOTHESIS

ALL THE ZEROS (NON-TRIVIAL ZEROS) IN $0 < \mathrm{Re}\,S < 1$ HAS $\mathrm{Re}\,S = \frac{1}{2}$.

$\underline{10^{13}}$ $\quad \mathrm{Im}(S) < 10^{13} \to 10^{14} \to 10^{15}$

$\zeta(S) = 0;\ \mathrm{Im}(S) < 10^{13} \Rightarrow \mathrm{Re}\,S = \frac{1}{2}$

\$1000 000 CLAY