Class 4: The Riemann ξ-function and the zeros of $\zeta(s)$

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Review:

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1. For $\mathrm{Re}\, s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

2. For $\mathrm{Re}\, s > 0$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t.$$

Let

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi x n^2}, \quad \text{for } x > 0.$$

Then, for $\mathrm{Re}\, s > 1$:

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \omega(x)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{\mathrm{d}x}{x}.$$

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It gives an analytic continuation (meromorphic) of $\zeta(s)$ in $\mathbb{C}$. In particular:

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The Riemann ξ-function is an entire function defined as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

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The functions $\xi(s)$ and $\zeta(s)$ have the same zeros in $0 < \mathrm{Re}\, s < 1$.

$\xi(s) \in \mathbb{R}$ and $\zeta(s) \in \mathbb{R}$ for $s \in \mathbb{R}$.

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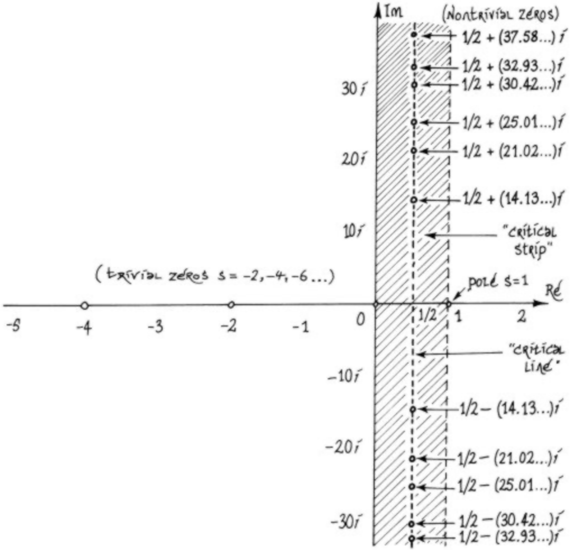
$$\xi(s) = \overline{\xi(\bar{s})} \text{ for all } s \in \mathbb{C}.$$


Im
(Nontrivial zeros)
1/2 + (37.58...) i
1/2 + (32.93...) i
1/2 + (30.42...) i
1/2 + (25.01...) i
1/2 + (21.02...) i
1/2 + (14.13...) i
"critical strip"
30 i
20 i
10 i
(trivial zeros s = -2, -4, -6 ...)
pole s=1
Re
-5
-4
-3
-2
-1
0
1/2
1
2
"critical line"
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-20 i
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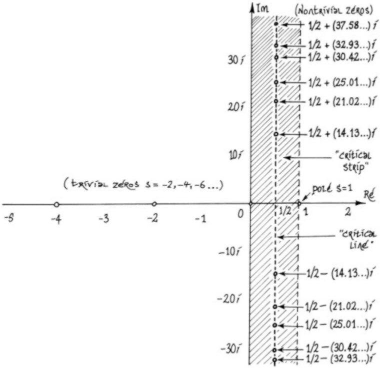
The Riemann Hypothesis - R.H.

Conjecture (18 November 1859)

All non-trivial zeros of $\zeta(s)$ have real part equal to $1/2$.

Theorem (Dave Platt and Tim Trudgian, 21 April 2020)

The Riemann Hypothesis is true up to height 3000175332800*. That is, the lowest* 12363153437138 *non-trivial zeros* ρ *have* $\mathrm{Re}\,\rho = 1/2$.



The order of $\xi(s)$

1. We say that f is an entire function of finite order if there is $\alpha > 0$ such that

$$|f(z)| \leq M\, e^{|z|^{\alpha}},$$

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4. The function ξ has order 1.

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4. $\log|\pi^{-s/2}| = \log \pi^{-\mathrm{Re}\, s/2} \ll |s|$.

We want to bound $\log \left| \Gamma\left(\frac{s}{2}\right) \right|$ in $|s| \geq 2$ and $\mathrm{Re}\, s \geq \frac{1}{2}$.

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Therefore ξ has order at most 1.

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5. $\log|\zeta(\sigma)| = O(\log\sigma)$.

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We conclude that

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Therefore ξ is an entire function of order 1.

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$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log N.$$

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Note that this sum is uniform convergent (absolutely convergent) in compacts.

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At the point $s = 0$ we have

$$B = -1 - \frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1) + \frac{\zeta'}{\zeta}(0).$$

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We know that

$$\sum_{\rho} \frac{1}{|\rho|} = \infty,$$

but with a certain order the sum (without modulus) converges. In fact, summing over

$$\mathrm{Re}\left\{\frac{1}{\rho}\right\} = \frac{1}{2}\left(\frac{1}{\rho} + \frac{1}{\overline{\rho}}\right) = \frac{\rho + \overline{\rho}}{2|\rho|^2} = \frac{\mathrm{Re}\,\rho}{|\rho|^2}.$$

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$$B+\mathrm{Re}\left\{\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)\right\}=-B-\mathrm{Re}\left\{\sum_{\rho}\left(\frac{1}{1-s-\rho}+\frac{1}{\rho}\right)\right\}.$$

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Since ρ is a zero if and only if $1-\rho$ is a zero. Then

$$\sum_{\rho} \mathrm{Re}\, \frac{1}{s-\rho} = \sum_{\rho} \mathrm{Re}\, \frac{1}{s-(1-\rho)} = -\sum_{\rho} \mathrm{Re}\, \frac{1}{1-s-\rho}.$$

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$$-2B = 2\sum_\rho \mathrm{Re}\left\{\frac{1}{\rho}\right\}.$$

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$$-B = \sum_{\gamma>0} \left(\mathrm{Re}\left\{\frac{1}{\beta + i\gamma}\right\} + \mathrm{Re}\left\{\frac{1}{\beta - i\gamma}\right\} \right) = \sum_{\gamma>0} \frac{2\beta}{\beta^2 + \gamma^2}.$$

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Fix $\rho = \beta_0 + i\gamma_0$ such that $\beta_0 \geq \frac{1}{2}$ and $\gamma_0 > 0$. Then

$$-\frac{B}{2} = \sum_{\gamma>0} \frac{\beta}{\beta^2 + \gamma^2} \geq \frac{\beta_0}{\beta_0^2 + \gamma_0^2} \geq \frac{1/2}{1 + \gamma_0^2}.$$

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