Class 5: Zero-free region

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Review:

Review:

The Riemann ξ-function is an entire function defined as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and $\xi(s) = \xi(1-s)$.

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   But, as $\sigma \to \infty$:

$$\frac{\sigma \log \sigma}{2} + O(\sigma) = \log|\xi(\sigma)| \leq M + |\sigma|.$$

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where γ is the Euler's constant.

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Note that this sum is uniform convergent (absolutely convergent) in compacts.

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In particular $\sum_\rho \mathrm{Re}\left\{\frac{1}{\rho}\right\}$ is absolutely convergent.

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Since ρ is a zero if and only if $1-\rho$ is a zero. Then

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$$-2B = 2\sum_\rho \operatorname{Re}\frac{1}{\rho}.$$

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Assume that $\rho = \beta + i\gamma$. Then

$$-B = \sum_{\gamma>0} \left(\mathrm{Re}\left\{\frac{1}{\beta + i\gamma}\right\} + \mathrm{Re}\left\{\frac{1}{\beta - i\gamma}\right\} \right) = \sum_{\gamma>0} \frac{2\beta}{\beta^2 + \gamma^2}.$$

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Fix $\rho = \beta_0 + i\gamma_0$ such that $\beta_0 \geq \frac{1}{2}$ and $\gamma_0 > 0$. Then

$$-\frac{B}{2} = \sum_{\gamma>0}\frac{\beta}{\beta^2+\gamma^2} \geq \frac{\beta_0}{\beta_0^2+\gamma_0^2} \geq \frac{1/2}{1+\gamma_0^2}.$$

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Theorem (Hadamard, de la Vallée -Poussin 1896)

For $t \in \mathbb{R}$, we have $\zeta(1 + it) \neq 0$.

First proof

We have uniformly in compacts of $\mathrm{Re}\, s > 1$:

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$$\log \zeta(\sigma + it) = \sum_p \sum_{k=1}^{\infty} \frac{1}{p^{k(\sigma+it)} k} = \sum_p \sum_{k=1}^{\infty} \frac{e^{-ikt \log p}}{p^{k\sigma} k}$$

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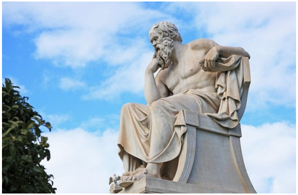

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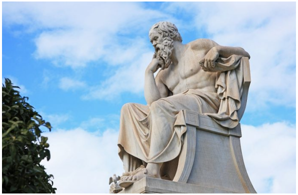

Recalling that $3 + 4\cos\theta + \cos 2\theta \geq 0$, we have

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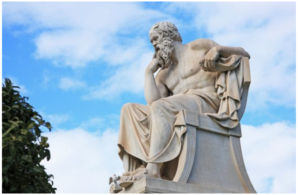

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for $\sigma > 1$ and $t \in \mathbb{R}$.

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Assume that $1 + it_0$ is a zero of $\zeta(s)$, then as $\sigma \to 1^+$:

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Taking derivative, we have

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Note that this series converges absolutely for $\mathrm{Re}\, s > 1$, because $|\Lambda(n)| \leq \log n$ and $\sum_n \frac{\log n}{n^\sigma} < \infty$ for $\sigma > 1$.

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$$-\frac{\zeta'}{\zeta}(\sigma) \leq B_1 + \frac{1}{\sigma-1}, \quad \sigma \to 1^+.$$

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Zero-free region

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Recall that, for $\sigma > 1$, $t \in \mathbb{R}$:

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

$$\mathrm{Re}\left\{ -\frac{\zeta'}{\zeta}(\sigma + it) \right\} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \cos(t \log n).$$

One can see that

$$\mathrm{Re}\left\{ -3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it) \right\}$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \Big(3 + 4\cos(t \log n) + \cos(2t \log n) \Big) \geq 0.$$

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Stirling's formula for the Gamma function:
For a fixed $\delta > 0$ and $-\pi + \delta < \arg(s) < \pi - \delta$,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1}),$$

as $|s| \to \infty$.

Let $s = \sigma + it$, $t \geq 2$ and $1 < \sigma \leq 2$:

$$\mathrm{Re} - \frac{\zeta'}{\zeta}(s) \leq C + \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) - \mathrm{Re}\sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

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$$\mathrm{Re} - \frac{\zeta'}{\zeta}(\sigma + 2it) \le C_1 \log(2t) - \sum_{\rho} \mathrm{Re}\,\frac{1}{s-\rho} - \sum_{\rho} \mathrm{Re}\,\frac{1}{\rho}.$$

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Summarize: Fix $\rho = \beta + it$ a zero of $\zeta(s)$. For $\sigma > 1$, close to 1 we have:

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$$\begin{aligned} 0 &\leq \mathrm{Re} \left\{ -3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it) \right\} \\ &\leq 2B_1 + \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + C_3 \log t \\ &\leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + D \log t \end{aligned}$$

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Choosing

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Choosing

$$\delta = \frac{1}{2D},$$

we have

$$\beta \le 1 - \frac{1}{14D \log t}.$$

There is $C > 0$ such that, if $\rho = \beta + it$ is a non-trivial zero of $\zeta(s)$, then

$$\beta \leq 1 - \frac{C}{\log t}.$$

There is $C > 0$ such that $\zeta(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{C}{\log t},$$

with $t \geq 2$.