Class 6: The function N(T)

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Review:

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The Riemann ξ-function is an entire function of order 1, defined as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and $\xi(s) = \xi(1-s)$.

The theory of entire functions give us:

1. Hadamard product:

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}.$$

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3. $\xi(s)$ has infinity zeros.
4.

$$\sum_{\rho}\frac{1}{|\rho|} = \infty.$$

Theorem (Hadamard, de la Vallée -Poussin 1896)

For $t \in \mathbb{R}$, we have $\zeta(1 + it) \neq 0$.

Theorem

There is $C > 0$ such that $\zeta(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{C}{\log t},$$

with $t \geq 2$.

Definition

Let $T \geq 2$ such that T is not the ordinate of a zero of $\zeta(s)$.
Let $N(T)$ be the number of non-trivial zeros of $\zeta(s)$ such that their imaginary parts are $< T$.

$$N(T) = \#\{\rho = \beta + i\gamma : 0 < \beta < 1,\, \zeta(\rho) = 0,\, 0 < \gamma < T\},$$

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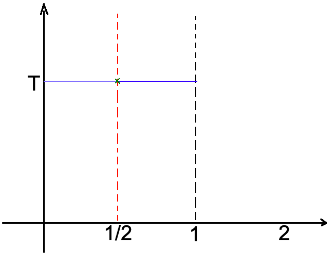

Definition

Let $T \geq 2$ such that T is not the ordinate of a zero of $\zeta(s)$. Define

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right),$$

where the argument is defined by the continuous variation along straight line segments joining the points 2, $2 + iT$ and $1/2 + iT$, with $\arg \zeta(2) = 0$.

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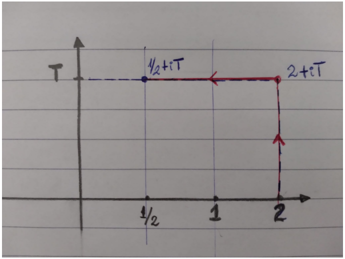
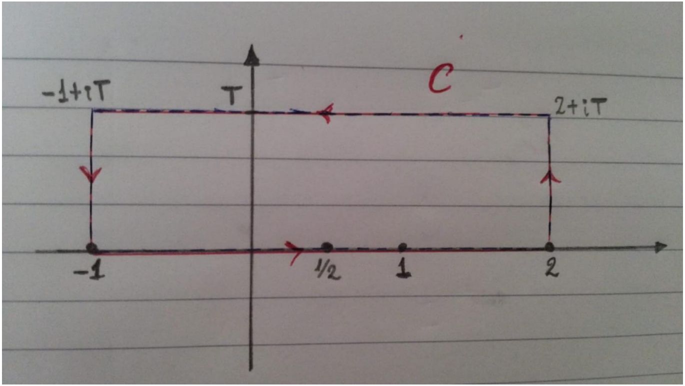
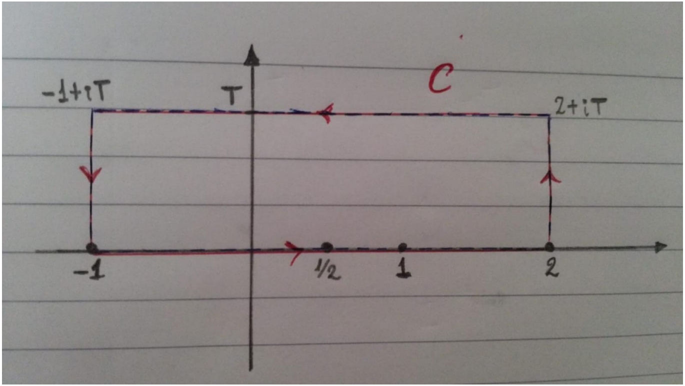


Proposition (Riemann-von Mangoldt Formula)

Let $T \geq 2$ such that T is not the ordinate of a zero of $\zeta(s)$. Then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right).$$


C
−1+iT
T
2+iT
−1
1/2
1
2



Argument principle

$$N(T) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\xi'(s)}{\xi(s)} \, \mathrm{d}s$$

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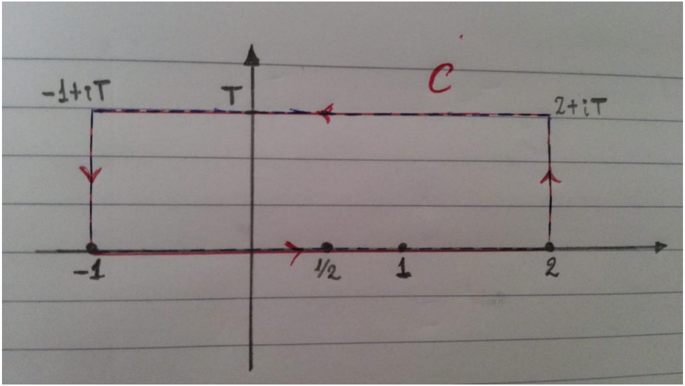

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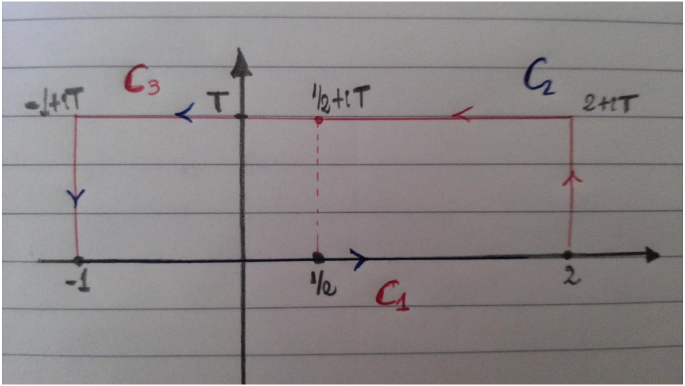

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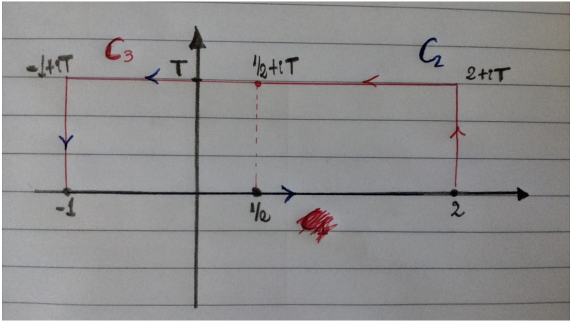
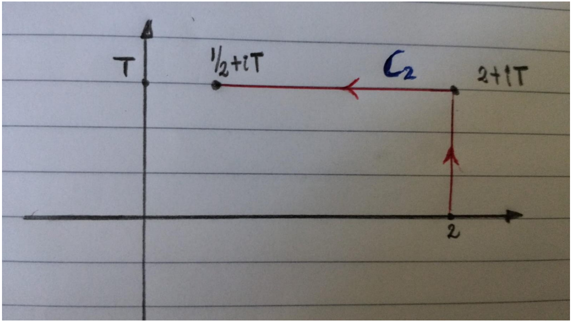
$$N(T) = \frac{1}{2\pi} \left(\mathrm{Im} \int_{\mathcal{C}_1} \frac{\xi'(s)}{\xi(s)} \, \mathrm{d}s + \mathrm{Im} \int_{\mathcal{C}_2} \frac{\xi'(s)}{\xi(s)} \, \mathrm{d}s + \mathrm{Im} \int_{\mathcal{C}_3} \frac{\xi'(s)}{\xi(s)} \, \mathrm{d}s \right)$$



$$N(T) = \frac{1}{2\pi}\left(\mathrm{Im}\int_{\mathcal{C}_2} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s + \mathrm{Im}\int_{\mathcal{C}_3} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s\right)$$

$\xi(s) = \xi(1-s) = \overline{\xi(1-\overline{s})}$, and $\xi'(s) = -\xi'(1-s) = -\overline{\xi'(1-\overline{s})}$ implies

$$\frac{\xi'(s)}{\xi(s)} = -\overline{\frac{\xi'(1-\overline{s})}{\xi(1-\overline{s})}}.$$



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Remember:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

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Logarithmic derivative of $\xi(s)$:

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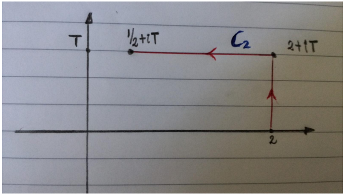
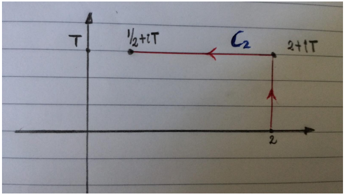
Logarithmic derivative of $\xi(s)$:

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s-1} - \frac{\log\pi}{2} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \frac{\zeta'(s)}{\zeta(s)}.$$



$$\pi N(T) = \operatorname{Im} \int_{\mathcal{C}_2} \frac{1}{s-1}\, \mathrm{d}s - \operatorname{Im} \int_{\mathcal{C}_2} \frac{\log \pi}{2}\, \mathrm{d}s + \operatorname{Im} \int_{\mathcal{C}_2} \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) \mathrm{d}s + \operatorname{Im} \int_{\mathcal{C}_2} \frac{\zeta'}{\zeta}(s)\, \mathrm{d}s.$$



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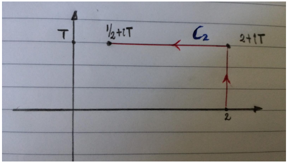
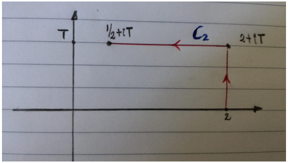
Note that

$$\mathrm{Im} \int_{\mathcal{C}_2} \frac{\zeta'}{\zeta}(s)\,\mathrm{d}s = \pi S(T).$$



1

$$\operatorname{Im}\int_{\mathcal{C}_2}\frac{1}{s-1}\,\mathrm{d}s=\frac{\pi}{2}+O\left(\frac{1}{T}\right).$$



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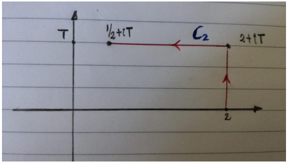

1

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2

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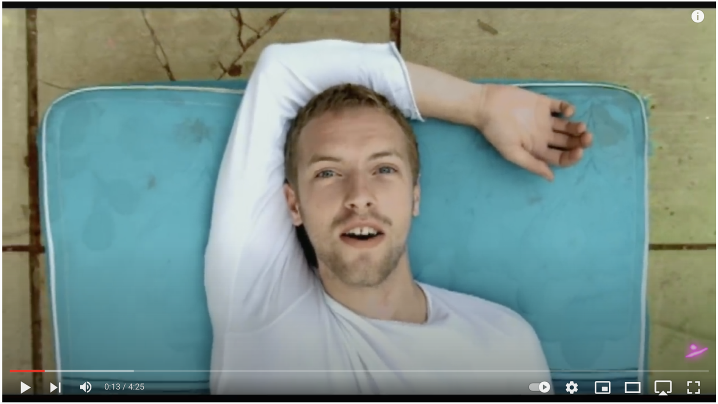
3

$$\mathrm{Im} \int_{\mathcal{C}_2} \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) \mathrm{d}s = \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} + \frac{3\pi}{8} + O\left(\frac{1}{T}\right).$$

Nobody said it was easy



#Coldplay #TheScientist #ARushofBloodtotheHead
Coldplay - The Scientist (Official Video)

$$\begin{aligned}\pi N(T) &= \operatorname{Im}\int_{\mathcal{C}_2}\frac{1}{s-1}\,\mathrm{d}s \,-\, \operatorname{Im}\int_{\mathcal{C}_2}\frac{\log\pi}{2}\,\mathrm{d}s \\ &\quad+\operatorname{Im}\int_{\mathcal{C}_2}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)\mathrm{d}s \,+\, \operatorname{Im}\int_{\mathcal{C}_2}\frac{\zeta'}{\zeta}(s)\,\mathrm{d}s \\ &= \frac{\pi}{2}-\frac{T}{2}\log\pi+\frac{T}{2}\log\frac{T}{2}-\frac{T}{2}+\frac{3\pi}{8}+O\left(\frac{1}{T}\right)+\pi S(T).\end{aligned}$$

$$\pi N(T) = \operatorname{Im} \int_{\mathcal{C}_2} \frac{1}{s-1}\,\mathrm{d}s - \operatorname{Im} \int_{\mathcal{C}_2} \frac{\log \pi}{2}\,\mathrm{d}s + \operatorname{Im} \int_{\mathcal{C}_2} \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)\mathrm{d}s + \operatorname{Im} \int_{\mathcal{C}_2} \frac{\zeta'}{\zeta}(s)\,\mathrm{d}s$$
$$= \frac{\pi}{2} - \frac{T}{2}\log \pi + \frac{T}{2}\log\frac{T}{2} - \frac{T}{2} + \frac{3\pi}{8} + O\left(\frac{1}{T}\right) + \pi S(T).$$

Proposition

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right).$$

If $T \geq 2$ is the ordinate of a zero of $\zeta(s)$, define

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2

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3.

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1

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2

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$$N(T) = \frac{T}{2\pi}\log T + O(T) = \frac{T}{2\pi}\log T\left(1 + O\left(\frac{1}{\log T}\right)\right).$$

$$N(T) \sim \frac{T}{2\pi}\log T, \text{ as } T \to \infty$$

Lemma

If $\rho = \beta + i\gamma$ *runs through the non-trivial zeros of* $\zeta(s)$ *then for* $T \geq 2$ *we have*

$$\sum_{\rho} \frac{1}{1 + (T - \gamma)^2} = O(\log T)$$

Let $s = \sigma + it$, $t \geq 2$ and $1 \leq \sigma \leq 2$:

$$\mathrm{Re} - \frac{\zeta'}{\zeta}(s) \leq C_1 \log t - \mathrm{Re} \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

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Then

$$\sum_\rho \mathrm{Re}\, \frac{1}{s-\rho} \leq C_1 \log t + \mathrm{Re}\, \frac{\zeta'}{\zeta}(s).$$

Let $s=\sigma+it$, $t\geq 2$ and $1\leq\sigma\leq 2$:

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Then

$$\sum_\rho\mathrm{Re}\,\frac{1}{s-\rho}\leq C_1\log t+\mathrm{Re}\,\frac{\zeta'}{\zeta}(s).$$

Leting $s=2+iT$ we have

$$\sum_\rho\mathrm{Re}\,\frac{1}{2+iT-\rho}\leq C_2\log T$$

$$\sum_{\rho} \mathrm{Re}\, \frac{1}{2 + iT - \rho} \leq C_2 \log T$$

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Letting $\rho = \beta + i\gamma$ $(0 < \beta < 1)$ we have

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Letting $\rho = \beta + i\gamma$ $(0 < \beta < 1)$ we have

$$\mathrm{Re} \frac{1}{2 + iT - \rho} = \mathrm{Re} \frac{1}{2 - \beta + i(T - \gamma)} = \frac{2 - \beta}{(2 - \beta)^2 + (T - \gamma)^2}$$

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$$\mathrm{Re}\, \frac{1}{2 + iT - \rho} \geq \frac{1}{4 + (T - \gamma)^2} \geq \frac{1}{4 + 4(T - \gamma)^2}$$

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$$\sum_{\rho} \frac{1}{4+4(T-\gamma)^2} \leq \sum_{\rho} \mathrm{Re}\, \frac{1}{2+iT-\rho} \leq C_2 \log T.$$

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Therefore

$$0 \leq \sum_{\rho} \frac{1}{1+(T-\gamma)^2} \leq 4C_2 \log t.$$

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Therefore

$$0 \leq \sum_{\rho} \frac{1}{1+(T-\gamma)^2} \leq 4C_2 \log t.$$

This implies that

$$\sum_{\rho} \frac{1}{1+(T-\gamma)^2} = O(\log T)$$

Corollary

For $T \geq 2$*, the number of non-trivial zeros of* $\zeta(s)$ *with* $T - 1 < \gamma < T + 1$ *is* $O(\log T)$.

Corollary

For $T \geq 2$, the number of non-trivial zeros of $\zeta(s)$ with $T-1 < \gamma < T+1$ is $O(\log T)$.

$$\sum_{\rho} \frac{1}{1+(T-\gamma)^2} = O(\log T)$$

Corollary

For $T \geq 2$*:*

$$\sum_{|\gamma - T| \geq 1} \frac{1}{(\gamma - T)^2} = O(\log T).$$