Class 7: The function S(T)

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Review:

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The Riemann ξ-function is an entire function of order 1, defined as:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and $\xi(s) = \xi(1-s)$.

The theory of entire functions give us:

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3. $\xi(s)$ has infinity zeros.

4. $$\sum_{\rho} \frac{1}{|\rho|} = \infty.$$

Theorem

There is $C > 0$ such that $\zeta(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{C}{\log t},$$

with $t \geq 2$.

Definition

Let $T \geq 2$. Let $N(T)$ be the number of non-trivial zeros of $\zeta(s)$ such that their imaginary parts are $< T$.

$$N(T) = \#\{\rho = \beta + i\gamma : 0 < \beta < 1, \, \zeta(\rho) = 0, \, 0 < \gamma \leq T\},$$

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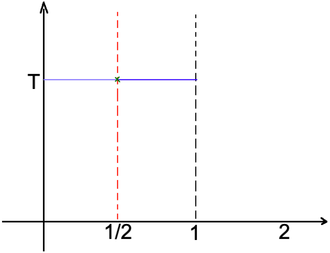

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Let $T \geq 2$ such that T is not the ordinate of a zero of $\zeta(s)$. Define

$$S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right),$$

where the argument is defined by the continuous variation along straight line segments joining the points 2, $2 + iT$ and $1/2 + iT$, with $\arg \zeta(2) = 0$. If T is the ordinate of a zero of $\zeta(s)$, define $S(T) = \text{lím}_{\varepsilon \to 0^+} S(T + \varepsilon)$

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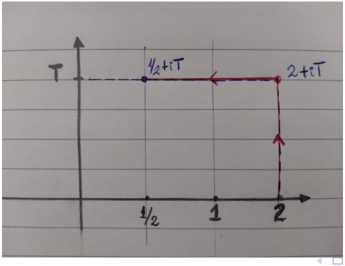

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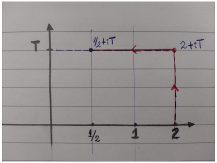
$$S(T) = \frac{1}{\pi} \mathrm{Im} \int_{\mathcal{C}_2} \frac{\zeta'}{\zeta}(s)\, \mathrm{d}s$$



1. Riemann-von Mangoldt Formula:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right).$$

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$$N(T) \sim \frac{T}{2\pi}\log T, \text{ as } T \to \infty$$

TODAY! 

$$S(T) = O(\log T)$$

Lemma

If $\rho = \beta + i\gamma$ runs through the non-trivial zeros of $\zeta(s)$ then for $T \geq 2$ we have

$$\sum_{\rho} \frac{1}{1 + (T - \gamma)^2} = O(\log T)$$

Let $s = \sigma + it$, $t \geq 2$ and $1 \leq \sigma \leq 2$:

$$\mathrm{Re} - \frac{\zeta'}{\zeta}(s) \leq C_1 \log t - \mathrm{Re} \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

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$$\sum_{\rho} \mathrm{Re} \frac{1}{s-\rho} \leq C_1 \log t + \mathrm{Re} \frac{\zeta'}{\zeta}(s).$$

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Then

$$\sum_\rho \mathrm{Re} \, \frac{1}{s-\rho} \leq C_1 \log t + \mathrm{Re} \, \frac{\zeta'}{\zeta}(s).$$

Leting $s = 2 + iT$ we have

$$\sum_\rho \mathrm{Re} \, \frac{1}{2+iT-\rho} \leq C_2 \log T$$

$$\sum_{\rho} \mathrm{Re}\, \frac{1}{2 + iT - \rho} \le C_2 \log T$$

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$$\mathrm{Re}\, \frac{1}{2 + iT - \rho} = \mathrm{Re}\, \frac{1}{2 - \beta + i(T - \gamma)} = \frac{2 - \beta}{(2 - \beta)^2 + (T - \gamma)^2}$$

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$$\mathrm{Re}\, \frac{1}{2+iT-\rho} \ge \frac{1}{4+(T-\gamma)^2} \ge \frac{1}{4+4(T-\gamma)^2}$$

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Therefore

$$0 \leq \sum_\rho \frac{1}{1+(T-\gamma)^2} \leq 4C_2 \log t.$$

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This implies that

$$\sum_{\rho} \frac{1}{1+(T-\gamma)^2} = O(\log T)$$

Corollary

For $T \geq 2$, the number of non-trivial zeros of $\zeta(s)$ with $T - 1 < \gamma < T + 1$ is $O(\log T)$.

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Corollary

For $T \geq 2$:

$$\sum_{|\gamma - T| \geq 1} \frac{1}{(\gamma - T)^2} = O(\log T).$$

Proposition

Let $s = \sigma + it$ such that $-1 \leq \sigma \leq 2$ and $t \geq 3$ and t is not the ordinate of a zero. Then

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log t),$$

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Logarithmic derivative of $\xi(s)$:

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$$\frac{\zeta'}{\zeta}(3+it) = -\frac{1}{2+it} + \frac{\log \pi}{2} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{5}{2}+\frac{it}{2}\right) + B + \sum_{\rho}\left(\frac{1}{3+it-\rho}+\frac{1}{\rho}\right).$$

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Now, for $-1 \le \sigma \le 2$ and $t \ge 3$ we have:

$$\begin{aligned}\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(3+it) = & -\frac{1}{\sigma - 1 + it} + \frac{1}{2+it} \\ & - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2} + 1 + \frac{it}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{5}{2} + \frac{it}{2}\right) \\ & + \sum_{\rho}\left(\frac{1}{\sigma + it - \rho} - \frac{1}{3 + it - \rho}\right).\end{aligned}$$

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We need to use our corollaries:

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho:|\gamma - t|\le 1} \left(\frac{1}{\sigma + it - \rho} - \frac{1}{3 + it - \rho} \right) + O(\log t).$$

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Theorem

For $T \geq 3$:

$$S(T) = O(\log T).$$

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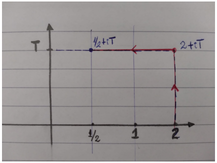

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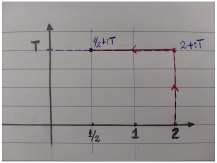

$$S(T) = \frac{1}{\pi} \operatorname{Im} \int_{[2,2+iT]} \frac{\zeta'}{\zeta}(s)\, \mathrm{d}s + \frac{1}{\pi} \operatorname{Im} \int_{[2+iT,1/2+iT]} \frac{\zeta'}{\zeta}(s)\, \mathrm{d}s$$

$$\frac{1}{\pi}\,\mathrm{Im}\int_{[2,2+iT]}\frac{\zeta'}{\zeta}(s)\,\mathrm{d}s=\frac{1}{\pi}\Big(\mathrm{Im}\,\log\zeta(2+iT)-\mathrm{Im}\,\log\zeta(2)\Big).$$

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Remember that

$$\log\zeta(s)=-\sum_{p}\log\left(1-\frac{1}{p^s}\right),$$

such that $\log\zeta(2)\in\mathbb{R}$.

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$$\log\zeta(2+it)=\sum_p\sum_{k=1}^{\infty}\frac{1}{p^{k(2+it)}k}.$$

Then $|\log\zeta(2+it)|\le\log\zeta(2)$.

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Then $|\log\zeta(2+it)|\leq\log\zeta(2)$. We conclude that

$$\frac{1}{\pi}\,\mathrm{Im}\int_{[2,2+iT]}\frac{\zeta'}{\zeta}(s)\,\mathrm{d}s=O(1).$$

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Note that

$$\operatorname{Im} \int_{[2+iT, 1/2+iT]} \frac{1}{s-\rho}\, \mathrm{d}s = \operatorname{Im} \left\{ \log(1/2 + iT - \rho) - \log(2 + iT - \rho) \right\}$$

and it is bounded by 2π.

$$\left| \sum_{|T-\gamma|\leq 1} \mathrm{Im} \int_{[2+iT,1/2+iT]} \frac{1}{s-\rho} \, \mathrm{d}s \right| \leq \sum_{|T-\gamma|\leq 1} 2\pi = O(\log T).$$

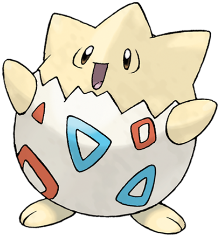
$$\left| \sum_{|T-\gamma|\le 1} \mathrm{Im} \int_{[2+iT,1/2+iT]} \frac{1}{s-\rho}\,\mathrm{d}s \right| \le \sum_{|T-\gamma|\le 1} 2\pi = O(\log T).$$

$$S(T) = \frac{1}{\pi} \sum_{|T-\gamma|\le 1} \mathrm{Im} \int_{[2+iT,1/2+iT]} \frac{1}{s-\rho}\,\mathrm{d}s + O(\log T)$$

$$\left| \sum_{|T-\gamma|\le 1} \mathrm{Im} \int_{[2+iT,1/2+iT]} \frac{1}{s-\rho}\,\mathrm{d}s \right| \le \sum_{|T-\gamma|\le 1} 2\pi = O(\log T).$$

$$S(T) = \frac{1}{\pi} \sum_{|T-\gamma|\le 1} \mathrm{Im} \int_{[2+iT,1/2+iT]} \frac{1}{s-\rho}\,\mathrm{d}s + O(\log T)$$

$$S(T) = O(\log T) + O(\log T).$$



$$S(T) = O(\log T)$$