Class 8: Approximation formula

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Review:

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1. For $\mathrm{Re}\, s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

2. For $\mathrm{Re}\, s > 0$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t.$$

For $\mathrm{Re}\, s > 1$ and $N \geq 1$, using integration by parts:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^s} &= \int_{1^-}^{N^+} \frac{1}{t^s} \mathrm{d}[t] \\
&= \frac{[N^+]}{N^s} - \frac{[1^-]}{1^s} + s \int_1^N \frac{[t]}{t^{s+1}} \mathrm{d}t \\
&= \frac{N}{N^s} + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t + s \int_1^N \frac{t}{t^{s+1}} \mathrm{d}t \\
&= N^{1-s} + s \int_1^N t^{-s} \mathrm{d}t + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t \\
&= N^{1-s} + s \left(\frac{N^{1-s}}{1-s} - \frac{1}{1-s} \right) + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t \\
&= \frac{N^{1-s}}{1-s} + 1 + \frac{1}{s-1} + s \int_1^N \frac{[t] - t}{t^{s+1}} \mathrm{d}t.
\end{aligned}$$

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Then, as $N \to \infty$, we have for $\mathrm{Re}\, s > 1$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s\int_1^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t.$$

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The function

$$s \mapsto \int_1^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t \text{ is an analytic function in } \mathrm{Re}\, s > 0,$$

Morera's theorem

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Also, in compacts of $\mathrm{Re}\, s > 0$,

$$\left| \int_1^N \frac{[t]-t}{t^{s+1}}\mathrm{d}t - \int_1^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t \right| \le \left| \int_N^\infty \frac{1}{t^{s+1}}\mathrm{d}t \right| \le \int_N^\infty \frac{1}{t^{\sigma_0+1}}\mathrm{d}t \to 0,$$

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Therefore, as $N \to \infty$

$$f_N(s) \to \int_1^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t,$$

uniformly in compacts of $\mathrm{Re}\, s > 0$.

For $\mathrm{Re}\, s > 0$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s\int_1^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t.$$

Theorem

Let $C > 1$ and $\sigma_0 > 0$. Then, for $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ and $|t| < \frac{2\pi x}{C}$.

For $\mathrm{Re}\, s > 1$ and $N \geq 2$, using integration by parts:

$$\begin{aligned}
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} &= \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s} \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \int_{N^+}^{\infty} \frac{1}{t^s} \mathrm{d}[t] \\
&= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{[t]}{t^s}\bigg|_{N^+}^{\infty} + s \int_{N}^{\infty} \frac{[t]}{t^{s+1}} \mathrm{d}t \\
&= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N}{N^s} + s \int_{N}^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t + s \int_{N}^{\infty} \frac{t}{t^{s+1}} \mathrm{d}t \\
&= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + s \int_{N}^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t.
\end{aligned}$$

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Note that the last integral defines an analytic function in $\mathrm{Re}\, s > 0$. Then, for $\mathrm{Re}\, s > 0$ and $N \geq 2$ we have:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + s \int_N^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t.$$

Clearly

$$s\int_N^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t = O\left(\frac{|s|}{\sigma_0 N^\sigma}\right).$$

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Therefore,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right).$$

Then, for $x \geq 1$ we have

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \sum_{x<n\leq N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right).$$

We want to explore the oscillatory sums

$$\sum_{x<n\leq N} \frac{1}{n^s} = \sum_{x<n\leq N} \frac{n^{-it}}{n^\sigma}.$$

Lemma

Let $F : [a, b] \to \mathbb{R}$ *be a differentiable function such that* $F'(x)$ *is monotonic, and* $F'(x) \geq m > 0$ *or* $F'(x) \leq -m < 0$ *in* $[a, b]$*. Then*

$$\left| \int_a^b e^{iF(x)} \mathrm{d}x \right| \leq \frac{8}{m}$$

Theorem (Second mean value theorem for integrals)

Let $G : [a, b] \to \mathbb{R}$ be a monotonic function and $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Then, there is $c \in (a, b]$ such that

$$\int_a^b G(x)f(x)\mathrm{d}x = G(a^+)\int_a^c f(x)\mathrm{d}x + G(b^-)\int_c^b f(x)\mathrm{d}x.$$

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$$\begin{aligned}
&\int_a^b \cos(F(x))\mathrm{d}x \\
&\quad = \int_a^b \frac{1}{F'(x)} \cos(F(x))F'(x)\mathrm{d}x \\
&\quad = \frac{1}{F'(a^+)} \int_a^c \cos(F(x))F'(x)\mathrm{d}x + \frac{1}{F'(b^-)} \int_c^b \cos(F(x))F'(x)\mathrm{d}x \\
&\quad = \frac{\sin(F(c)) - \sin(F(a))}{F'(a^+)} + \frac{\sin(F(c)) - \sin(F(a))}{F'(b^-)}.
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We conclude that

$$\left| \int_a^b e^{iF(x)}\mathrm{d}x \right| \leq \frac{8}{m}.$$

Lemma

Let $F : [a, b] \to \mathbb{R}$ *be a twice continuously differentiable function such that* $F'(x)$ *is monotonic, and* $F'(x) \geq m > 0$ *or* $F'(x) \leq -m < 0$ *in* $[a, b]$. *Then*

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