Class 9: Approximation formula II

Andrés Chirre
Norwegian University of Science and Technology - NTNU

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Review:

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1. For $\mathrm{Re}\, s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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2. For $\operatorname{Re} s > 0$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s\int_1^{\infty} \frac{[t]-t}{t^{s+1}} \mathrm{d}t.$$

Theorem (Approximation formula)

Let $C > 1$ and $\sigma_0 > 0$. Then, for $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ and $|t| < \frac{2\pi x}{C}$.

For $\mathrm{Re}\, s > 1$ and $N \geq 2$, using integration by parts in:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s}.$$

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Note that the last integral defines an analytic function in $\operatorname{Re} s > 0$. Then, for $\operatorname{Re} s > 0$ and $N \geq 2$ we have:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + s\int_N^\infty \frac{[t]-t}{t^{s+1}}\mathrm{d}t.$$

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Clearly

$$s \int_N^\infty \frac{[t]-t}{t^{s+1}} \mathrm{d}t = O\left(\frac{|s|}{\sigma_0 N^\sigma} \right).$$

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Then, for $x \geq 1$ we have for $N \geq x$:

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \sum_{x < n \leq N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right).$$

We want to explore the oscillatory sums

$$\sum_{x<n\le N}\frac{1}{n^s}=\sum_{x<n\le N}\frac{n^{-it}}{n^\sigma}=\sum_{x<n\le N}\frac{e^{-it\log n}}{n^\sigma}.$$

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$$\zeta(s)=\sum_{n\leq x}\frac{1}{n^s}+\sum_{x<n\leq N}\frac{e^{-it\log n}}{n^\sigma}-\frac{N^{1-s}}{1-s}+O\left(\frac{|s|}{N^\sigma}\right).$$

Lemma

Let $F : [a, b] \to \mathbb{R}$ *be a differentiable function such that* $F'(x)$ *is monotonic, and* $F'(x) \geq m > 0$ *or* $F'(x) \leq -m < 0$ *in* $[a, b]$*. Then*

$$\left| \int_a^b e^{iF(x)} \mathrm{d}x \right| \leq \frac{8}{m}$$

Theorem (Second mean value theorem for integrals)

Let $G : [a, b] \to \mathbb{R}$ be a monotonic function and $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Then, there is $c \in (a, b]$ such that

$$\int_a^b G(x)f(x)\mathrm{d}x = G(a^+)\int_a^c f(x)\mathrm{d}x + G(b^-)\int_c^b f(x)\mathrm{d}x.$$

Proposition

Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f'(x)$ is monotone and $|f'(x)| \le \delta < 1$. Then

$$\sum_{a<n\le b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} \mathrm{d}x + O\left(\frac{1}{1-\delta}\right).$$

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We can assume that $f'(x)$ is increasing.

$$\sum_{a<n\le b} e^{2\pi i f(n)} = \int_{a^+}^{b^+} e^{2\pi i f(x)} \mathrm{d}[x]$$

$$= e^{2\pi i f(x)}[x]\Big|_{a^+}^{b^+} - \int_a^b 2\pi i f'(x) e^{2\pi i f(x)}[x]\mathrm{d}x$$

$$= e^{2\pi i f(x)}[x]\Big|_a^b + \int_a^b 2\pi i f'(x) e^{2\pi i f(x)}(x-[x]-\tfrac{1}{2})\mathrm{d}x$$

$$- \int_a^b 2\pi i f'(x) e^{2\pi i f(x)}(x-\tfrac{1}{2})\mathrm{d}x$$

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$$
\begin{aligned}
\sum_{a<n\leq b} e^{2\pi i f(n)} &= e^{2\pi i f(x)}[x]\Big|_a^b + \int_a^b 2\pi i f'(x) e^{2\pi i f(x)} (x-[x]-\tfrac{1}{2})\mathrm{d}x \\
&\quad - \int_a^b \left\{e^{2\pi i f(x)}\right\}' (x-\tfrac{1}{2})\mathrm{d}x \\
&= e^{2\pi i f(x)}[x]\Big|_a^b + \int_a^b 2\pi i f'(x) e^{2\pi i f(x)} (x-[x]-\tfrac{1}{2})\mathrm{d}x \\
&\quad - \left(e^{2\pi i f(x)}(x-\tfrac{1}{2})\Big|_a^b - \int_a^b e^{2\pi i f(x)}\mathrm{d}x \right) \\
&= e^{2\pi i f(x)}([x]-x+\tfrac{1}{2})\Big|_a^b + \int_a^b e^{2\pi i f(x)}\mathrm{d}x \\
&\quad + \int_a^b 2\pi i f'(x) e^{2\pi i f(x)} (x-[x]-\tfrac{1}{2})\mathrm{d}x.
\end{aligned}
$$

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = e^{2\pi i f(x)}([x] - x - \tfrac{1}{2})\Big|_a^b + \int_a^b e^{2\pi i f(x)}\mathrm{d}x$$

$$+ \int_a^b 2\pi i f'(x) e^{2\pi i f(x)}(x - [x] - \tfrac{1}{2})\mathrm{d}x$$

$$= \int_a^b e^{2\pi i f(x)}\mathrm{d}x + \int_a^b 2\pi i f'(x) e^{2\pi i f(x)}(x - [x] - \tfrac{1}{2})\mathrm{d}x$$

$$+ O(1).$$

$$\sum_{a<n\leq b} e^{2\pi i\, f(n)} = \int_a^b e^{2\pi i\, f(x)} \mathrm{d}x + \int_a^b 2\pi i\, f'(x) e^{2\pi i\, f(x)} (x - [x] - \tfrac{1}{2}) \mathrm{d}x + O(1)$$

Sawtooth function

$$\psi(x) = \begin{cases} x - [x] - \frac{1}{2}, & \textit{if } x \notin \mathbb{Z} \\ 0 & \textit{if } x \in \mathbb{Z}. \end{cases}$$

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We know that for all $x \in \mathbb{R}$:

$$\psi(x) = -\frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{m},$$

and the partial sums of its Fourier series are uniformly bounded.

$$\begin{aligned}
\int_a^b & 2\pi i\, f'(x) e^{2\pi i\, f(x)} (x - [x] - \tfrac{1}{2}) \mathrm{d}x \\
&= -\int_a^b 2\pi i\, f'(x) e^{2\pi i\, f(x)} \left(\frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2\pi m x}{m} \right) \mathrm{d}x \\
&= -\sum_{m=1}^{\infty} \int_a^b 2i\, f'(x) e^{2\pi i\, f(x)}\ \frac{\sin 2\pi m x}{m} \mathrm{d}x \qquad \text{(DCT)} \\
&= -\sum_{m=1}^{\infty} \frac{1}{m} \int_a^b f'(x) e^{2\pi i\, f(x)} \big(e^{2\pi i m x} - e^{-2\pi i m x} \big) \mathrm{d}x.
\end{aligned}$$

$$\sum_{a<n\le b} e^{2\pi i\, f(n)} = \int_a^b e^{2\pi i\, f(x)}\mathrm{d}x + \int_a^b 2\pi i\, f'(x) e^{2\pi i\, f(x)}(x-[x]-\tfrac{1}{2})\mathrm{d}x + O(1)$$

$$= \int_a^b e^{2\pi i\, f(x)}\mathrm{d}x + O(1)$$

$$-\sum_{m=1}^{\infty}\frac{1}{m}\int_a^b f'(x) e^{2\pi i\, f(x)}\left(e^{2\pi i m x} - e^{-2\pi i m x}\right)\mathrm{d}x.$$

Lemma

Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f'(x)$ is increasing and $|f'(x)| \le \delta < 1$. Then

$$\sum_{m=1}^{\infty} \left| \frac{1}{m} \int_a^b f'(x) e^{2\pi i (f(x)+mx)} \right| \mathrm{d}x = O\left(\frac{1}{1-\delta}\right),$$

and

$$\sum_{m=1}^{\infty} \left| \frac{1}{m} \int_a^b f'(x) e^{2\pi i (f(x)-mx)} \right| \mathrm{d}x = O\left(\frac{1}{1-\delta}\right).$$

$$\sum_{m=1}^{\infty} \left| \frac{1}{m} \int_a^b f'(x) e^{2\pi i (f(x)+mx)} \right| \mathrm{d}x$$

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$$\int_a^b f'(x)\cos(2\pi(f(x)+mx))\mathrm{d}x = \frac{1}{2\pi}\int_a^b \frac{f'(x)}{f'(x)+m}\big(\sin(2\pi(f(x)+mx))\big)'\mathrm{d}x$$

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Therefore:

$$\sum_{m=1}^{\infty}\left|\frac{1}{m}\int_a^b f'(x)e^{2\pi i(f(x)+mx)}\right|\mathrm{d}x \ll \sum_{m=1}^{\infty}\frac{\delta}{m\,(-\delta+m)} = O\left(\frac{1}{1-\delta}\right)$$

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$$\sum_{m=1}^{\infty}\left|\frac{1}{m}\int_a^b f'(x)e^{2\pi i(f(x)-mx)}\right|\mathrm{d}x \ll \sum_{m=1}^{\infty}\frac{\delta}{m\,(m-\delta)} \ll \frac{1}{1-\delta}.$$

Proposition

Let $f : [a, b] \to \mathbb{R}$ *be a continuously differentiable function such that* $f'(x)$ *is monotone and* $|f'(x)| \leq \delta < 1$. *Then*

$$\sum_{a<n\leq b} e^{2\pi i\, f(n)} = \int_a^b e^{2\pi i\, f(x)} \mathrm{d}x + O\left(\frac{1}{1-\delta}\right).$$

Proposition

Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f'(x)$ is monotone and $|f'(x)| \leq \delta < 1$. Let $g : [a, b] \to \mathbb{R}$ be a continuously differentiable, positive and decreasing function. Then

$$\sum_{a<n\leq b} g(n)e^{2\pi i\, f(n)} = \int_a^b g(u)e^{2\pi i\, f(u)}\mathrm{d}u + O\left(\frac{g(a)}{1-\beta}\right).$$

Let $C > 1$ and $\sigma_0 > 0$. For $s = \sigma + it$ with $0 < \sigma \leq \sigma_0 \leq 1$ and $x \geq 1$ we have for $N > x$:

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \sum_{x < n \leq N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right).$$

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$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \sum_{x < n \leq N} \frac{e^{2\pi i(\frac{-t \log n}{2\pi})}}{n^\sigma} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right).$$

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Assume that $0 < t < \frac{2\pi x}{C}$. Define the functions $f, g : [x, N] \to \mathbb{R}$

$$f(u) = -\frac{t \log u}{2\pi} \quad \text{and} \quad g(u) = \frac{1}{u^\sigma}.$$

Note that $f'(u) = -\frac{t}{2\pi u}$, and $|f'(x)| < \frac{1}{C}$.

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Note that $f'(u) = -\frac{t}{2\pi u}$, and $|f'(x)| < \frac{1}{C}$.
These functions satisfy our conditions in the last proposition. Then:

$$\sum_{x < n \le N} \frac{e^{2\pi i(\frac{-t \log n}{2\pi})}}{n^\sigma} = \int_x^N \frac{1}{x^s}\,\mathrm{d}x + O\left(\frac{x^{-\sigma}}{1 - 1/C}\right).$$

$$\sum_{x<n\leq N} \frac{e^{2\pi i(\frac{-t\log n}{2\pi})}}{n^{\sigma}} = \int_x^N \frac{1}{x^s}\,\mathrm{d}x + O_{C,\sigma_0}(x^{-\sigma}).$$

$$\sum_{x<n\le N} \frac{e^{2\pi i\left(\frac{-t\log n}{2\pi}\right)}}{n^\sigma} = \int_x^N \frac{1}{x^s}\,\mathrm{d}x + O_{C,\sigma_0}(x^{-\sigma}).$$

Then

$$\zeta(s) = \sum_{n\le x} \frac{1}{n^s} + \int_x^N \frac{1}{x^s}\,\mathrm{d}x + O_{C,\sigma_0}(x^{-\sigma}) - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right).$$

$$\sum_{x<n\leq N} \frac{e^{2\pi i\left(\frac{-t\log n}{2\pi}\right)}}{n^{\sigma}} = \int_x^N \frac{1}{x^s}\,\mathrm{d}x + O_{C,\sigma_0}(x^{-\sigma}).$$

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This implies

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{C,\sigma_0}(x^{-\sigma}) + O\left(\frac{|s|}{N^{\sigma}}\right).$$

$$\sum_{x<n\leq N} \frac{e^{2\pi i(\frac{-t\log n}{2\pi})}}{n^{\sigma}} = \int_x^N \frac{1}{x^s}\,\mathrm{d}x + O_{C,\sigma_0}(x^{-\sigma}).$$

Then

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Letting $N \to \infty$ we conclude.