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More than One Third of Zeros of Riemann's Zeta-Function are on $\sigma = 1/2$*

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1. INTRODUCTION

An historical account as well as the techniques used to prove the existence of zeros of the Riemann zeta-function, $\zeta(s)$, $s = \sigma + it$, on $\sigma = 1/2$ appears in [1, Chap. X]. Let $N_0(T)$ be the number of zeros of $\zeta(1/2 + it)$ on $0 < t \leqslant T$. The best result to date is due to Selberg [2] who showed that there is an effectively computable positive constant c such that

$$N_0(T) > cN(T),$$

where

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

is the number of zeros of the zeta-function in $0 \leqslant \sigma \leqslant 1$, $0 < t \leqslant T$. Selberg's proof actually goes further and proves the result in $(T, T + U)$ for suitable U. Selberg's proof involved combining a very effective non-negative mollifier to compensate for irregularities in the size of $|\zeta(1/2 + it)|$ with the method of Hardy and Littlewood [1, §10.7; 3].

Here it will be proved by a different method that

$$N_0(T) > 1/3N(T).$$

The method will depend on the fact that the argument of an appropriate function changes sufficiently rapidly. A device of this kind was used by Siegel [4] on the function $h(s) f_1(s)$, described later, which occurs in the Riemann–Siegel formula, (2.7), to get the Hardy–Littlewood result that $N_0(T) > CT$. Siegel obtained a definite value for C. It appears to

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me that the function $f_1(s)$ is not amenable to improvement with a mollifier.

It will be convenient to let

$$L = \log(T/2\pi).$$

THEOREM. *Let $U = T/L^{10}$. Then*

$$N_0(T+U) - N_0(T) > \tfrac{1}{3}[N(T+U) - N(T)].$$

An announcement of this result appeared in [5].

A consequence of the proof pointed out to me by H. L, Montgomery is the following.

COROLLARY. *Let m denote the multiplicity of a zero of the zeta-function on $\sigma = 1/2$. Then*

$$\sum (m-1) < \frac{.33U}{2\pi} \log \frac{T}{2\pi},$$

where $\sum$ is over the zeros in the interval from $1/2 + iT$ to $1/2 + i(T+U)$. From this follows at once

$$\sum_{m \geqslant 2} m < \frac{.66U}{2\pi} \log \frac{T}{2\pi}.$$

The basic idea of the proof can be developed quickly although the subsequent details are lengthy. Let $h(s) = \pi^{-s/2}\,\pi(s/2)$. Then

$$h(s)\,\zeta(s) = h(1-s)\,\zeta(1-s). \tag{1.1}$$

By Stirling's formula $h(s) = \exp f(s)$, where

$$f(s) = \tfrac{1}{2}(s-1)\log\frac{s}{2\pi} - \frac{s}{2} + C_0 + O\left(\frac{1}{s}\right) \tag{1.2}$$

for $|\arg s| \leqslant \pi - \delta$ and $|\operatorname{Im}\log(s/2\pi)| < \pi$. Differentiation yields

$$\frac{h'}{h}(s) = f'(s) = \tfrac{1}{2}\log\frac{s}{2\pi} + O\left(\frac{1}{s}\right). \tag{1.3}$$

For $|\sigma| < 10$ and large t, it follows that

$$f'(s) + f'(1-s) = \log(t/2\pi) + O(1/t). \tag{1.4}$$

Taking the derivative of (1.1) and using (1.1) to eliminate $\zeta(1-s)$ yields

$$h(s)\,\zeta(s)[f'(s) + f'(1-s)] = -h(s)\,\zeta'(s) - h(1-s)\,\zeta'(1-s). \tag{1.5}$$

On $s = 1/2 + it$ the right side of (1.5) is the sum of two complex conjugate, and zeros of the right side will occur where

$$\arg[h\zeta'(1/2 + it)] = \pi/2(\bmod \pi). \tag{1.6}$$

Taking account of (1.4), it follows that the zeros due to (1.6) are those of $h\zeta(1/2 + it)$ on the left side of (1.5). Since h is never zero, they are the zeros of $\zeta(1/2 + it)$ itself. Thus where (1.6) holds there occurr the zeros of $\zeta(1/2 + it)$.

By (1.1), if $\chi(s) = h(1 - s)/h(s)$, then $\zeta(s) = \chi(s)\zeta(1 - s)$ and so

$$\zeta'(s) = -\chi(s)\{[f'(s) + f'(1 - s)]\,\zeta(1 - s) + \zeta'(1 - s)\}. \tag{1.7}$$

Thus by (1.6) the zeros of $\zeta(1/2 + it)$ occur where

$$\arg(h(1 - s)\{[f'(s) + f'(1 - s)]\,\zeta(1 - s) + \zeta'(1 - s)\}) \equiv \pi/2(\bmod \pi)$$

on $\sigma = 1/2$ or, what is the same thing, where

$$\arg(h(s)\{[f'(s) + f'(1 - s)]\,\zeta(s) + \zeta'(s)\}) \equiv \pi/2(\bmod \pi) \tag{1.8}$$

on $\sigma = 1/2$. But $\arg h(s)$ is available from (1.2). Because of this and (1.4), it suffices in determining how frequently (1.8) holds to find the change in the argument of

$$G(s) = \zeta(s) + \zeta'(s)/[f'(s) + f'(1 - s)] \tag{1.9}$$

on the 1/2-line. Indeed, if $\arg G(1/2 + it)$ did not change, it would follow from (1.8) and Stirling's formula (1.2) that $\zeta(1/2 + it)$ would have essentially its full quota of zeros, $N_0(T) = N(T) + O(\log T)$. What will be shown here is that $\arg G(1/2 + it)$ is sufficiently restricted so that the Theorem can be proved.

Let D be the closed rectangle with vertices at $1/2 + iT$, $3 + iT$, $3 + i(T + U)$, $1/2 + i(T + U)$. Putting aside for the present the complication of zeros of $G(s)$ on the boundary, the change in $\arg G(s)$ around D is 2π times the number of zeros of G in D, $N_G(D)$. On the right side D

$$\begin{aligned} |\,G(3 + it) - 1\,| &\leqslant \sum_2^\infty n^{-3} + O(1/L) \\ &\leqslant \frac{1}{8} + \int_2^\infty \frac{dv}{v^3} + O(1/L) < 1/3 \end{aligned} \tag{1.10}$$

for large t. Hence arg G changes by less than π. On the lower side of D, Jensen's theorem is used in a familiar way [1, §9.4] on

$$G(iT + w) + G(-iT + w) \tag{1.11}$$

in a circle of radius 3 with center at $w = 3$ to show that on this side $\arg G = O(L)$. A similar result holds in the upper edge. Thus

$$\arg G(1/2 + it)]_T^{T+U} = -2\pi N_G(D) + O(L). \tag{1.12}$$

It is important therefore to find an upper bound for $N_G(D)$, the number of zeros of G in D, in order to find the change in arg G on the 1/2-line.

2. An Integral Associated with the Number of Zeros of G in D

An upper bound for $N_G(D)$ is found in a familiar way using a lemma of Littlewood [1, §9.9]. Let a be a function of T to be specified precisely later. For the present, it suffices to take $0 < a < 1/2$ and

$$1/2 - a = O(1/L). \tag{2.1}$$

Let D_1 be the closed rectangle with vertices $a + iT$, $3 + iT$, $3 + i(T + U)$, $a + i(T + U)$. Let $F(s)$ be analytic in D_1. Suppose $F(3 + it) \neq 0$. Determine $\arg F(\sigma + iT)$ by continuation left from $3 + iT$. If a zero is reached on the lower edge, use $\lim F(\sigma + iT - i\epsilon)$ as $\epsilon \to +0$ and $\lim F(\sigma + i(T + U + \epsilon))$ on the upper edge. Make horizontal cuts in D_1 from the left side to the zeros of F in D_1. Take $\int \log F(s)$ around the contour consisting of D_1 and the cuts to get

$$\begin{aligned} &\int_T^{T+U} \log |F(a + it)| \, dt - \int_T^{T+U} \log |F(3 + it)| \, dt \\ &\qquad + \int_a^3 \arg F(\sigma + i(T + U)) \, d\sigma - \int_a^3 \arg F(\sigma + iT) \, d\sigma \\ &= 2\pi \sum \text{dist}, \end{aligned} \tag{2.2}$$

where $\sum$ dist is the sum of the distances of the zeros of F in D_1 from the left side. This is Littlewood's lemma. It will be applied not to G but to ψG, where ψ is a mollifier the rationale of which will be explained later. Let $y = T^{1/2}/L^{20}$. Then

$$\psi(s) = \sum b_j/j^s, \tag{2.3}$$

where $\sum$ is for $1 \leqslant j \leqslant y$ and

$$b_j = \frac{\mu(j) \log y/j}{j^{1/2-a} \log y}, \tag{2.4}$$

where μ is the Mobius function. It was already shown in connection with (1.11) that $\arg G(\sigma + iT)$ and $\arg G(\sigma + i(T + U)) = O(L)$. Similar reasoning applies to ψ. Hence

$$\int_a^3 \arg(\psi G(\sigma + iT))\, d\sigma = O(L)$$

and similarly at $T + U$. From (1.9),

$$\int_T^{T+U} \log G(3 + it)\, dt = \int_T^{T+U} \log \zeta(3 + it)\, dt + O(U/L).$$

Since for $\sigma > 1$

$$\log \zeta(s) = -\sum \frac{\Lambda(n)}{n^s \log n},$$

it follows taking the real part that

$$\int_T^{T+U} \log |G(3 + it)|\, dt = O(U/L).$$

For the entire function $\psi(s)$, let

$$\psi(s) = 1 + \psi_1(s).$$

Then for $\sigma \geqslant 3$, since $| b_j | \leqslant 1$,

$$|\psi_1(s)| \leqslant \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \int_3^\infty \frac{dv}{v^\sigma} \leqslant \frac{1}{2^\sigma} + \frac{5}{2}\frac{1}{3^\sigma} < \frac{2}{2^\sigma}.$$

Therefore, $\log \psi(s)$ is analytic for $\sigma \geqslant 3$. Integrating on the contour $\sigma + iT$, $3 \leqslant \sigma < \infty$; $3 + it$, $T \leqslant t \leqslant T + U$; $\sigma + i(T + U)$, $3 \leqslant \sigma < \infty$ gives

$$\left| \int_T^{T+U} \log \psi(3 + it)\, dt \right| \leqslant 8 \int_3^\infty \frac{d\sigma}{2^\sigma} = O(1)$$

and so

$$\int_T^{T+U} \log |\psi(3 + it)|\, dt = O(1).$$

Taking $F = \psi G$ in (2.2) and using all of the above,

$$\int_T^{T+U} \log | \psi G(a + it)| \, dt + O(U/L) = 2\pi \sum \text{dist},$$

where here the distances are to the zeros of ψG in D_1. These zeros include those of G itself in the closed rectangle D, $N_G(D)$. These latter are at least distance $1/2 - a$ from $\sigma = a$. Hence

$$\int_T^{T+U} \log | \psi G(a + it)| \, dt + O(U/L) \geqslant 2\pi(1/2 - a) \, N_G(D). \tag{2.5}$$

Using the concavity of the logarithm,

$$\int_T^{T+U} \log | \psi G(a + it)| \, dt = 1/2 \int_T^{T+U} \log | \psi G(a + it)|^2 \, dt$$
$$\leqslant \tfrac{1}{2} U \log \left(\frac{1}{U} \int_T^{T+U} | \psi G(a + it)|^2 \, dt \right). \tag{2.6}$$

The role of ψ is to make the last inequality sharper since $| \psi G |^2$ is flatter than $| G |^2$ itself.

To compute the last integral it is necessary to express $G(s)$ by the Riemann–Siegel formula [4; 1, §2.10; 6] which is

$$\zeta(s) = f_1(s) + \chi(s) f_2(s), \tag{2.7}$$

where

$$f_1(s) = \frac{1}{2i} \int_C \frac{e^{\pi i w^2} w^{-s}}{\sin \pi w} \, dw, \tag{2.8}$$

where C is the line of slope 1 through $w = 1/2$ and with Im w decreasing. Similarly,

$$f_2(s) = \frac{1}{2i} \int_{\bar{C}} \frac{e^{-\pi i w^2} w^{s-1}}{\sin \pi w} \, dw, \tag{2.9}$$

where $\bar{C}$ has slope -1, passes through $w = 1/2$, and has Im w decreasing From the derivative of (2.7) follows

$$G(s) = f_1(s) + (f_1'(s) + \chi f_2'(s))/(f'(s) + f'(1 - s)). \tag{2.10}$$

Deforming the contour of (2.8), for $| \sigma | < 10$, so that it goes through the saddle point, essentially at $w = (t/2\pi)^{1/2}$, yields [4]

$$f_1(s) = g_1(s) + O(t^{-\sigma/2}),$$

where

$$g_1(s) = \sum_{n \leqslant (t/2\pi)^{1/2}} n^{-s} \tag{2.11}$$

as is familiar from the approximate functional equation. Similarly for the derivatives, one finds

$$f_1'(s) = -g_2(s) + O(t^{-\sigma/2} \log t),$$
$$f_2'(s) = g_3(s) + O(t^{(\sigma-1)/2} \log t),$$

where

$$g_2(s) = \sum_{n \leqslant (t/2\pi)^{1/2}} n^{-s} \log n, \tag{2.12}$$

$$g_3(s) = \sum_{n \leqslant (t/2\pi)^{1/2}} n^{s-1} \log n. \tag{2.13}$$

It is a consequence of Stirling's formula that [1, §4.12.3] for $|\sigma| < 10$,

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{1/2-\sigma} \exp\left[\frac{\pi i}{4} - it \log \frac{t}{2\pi e}\right]\left(1 + O\left(\frac{1}{t}\right)\right). \tag{2.14}$$

Note x_1 depends on σ where

$$\chi_1(t) = \left(\frac{t}{2\pi}\right)^{1/2-\sigma} \exp\left[\frac{\pi i}{4} - it \log \frac{t}{2\pi e}\right]. \tag{2.15}$$

Then because of (2.1), (1.4) and the above yields in (2.10)

$$G(a + it) = H(a + it) + H_1(t), \tag{2.16}$$

where

$$H(s) = g_1(s) + [-g_2(s) + \chi_1(t)\, g_3(s)]/\log(t/2\pi) \tag{2.17}$$

and

$$\begin{aligned} H_1(t) &= O(t^{-1/4}) + (|\, g_2(a + it)| + |\, g_3(a + it)|)\, O(t^{-1}) \\ &= O(t^{-1/4}), \end{aligned} \tag{2.18}$$

since by (2.12), $|\, g_2(a + it)| = O(t^{1/4} \log t)$ and similarly for $|\, g_3\,|$. From (2.16),

$$\begin{aligned} &\int_T^{T+U} |\, \psi G(a + it)|^2\, dt \\ &\quad = \int_T^{T+U} |\, \psi H(a + it)|^2\, dt + O\left(\int_T^{T+U} |\, H_1(t)\, \psi(a + it)|^2\, dt\right) \\ &\quad\quad + \left(\int_T^{T+U} |\, \psi H(a + it)|^2\, dt\right)^{1/2} O\left(\left(\int_T^{T+U} |\, H_1(t)\, \psi(a + it)|^2\, dt\right)^{1/2}\right). \end{aligned} \tag{2.19}$$

From (2.3) and (2.4) follows in a standard way that

$$\int_T^{T+U} |\psi(a+it)|^2\,dt = U\sum b_j^2/j^{2a} + O(T^{1/2}\log T)$$
$$= O(UL).$$

From this and (2.18),

$$\int_T^{T+U} |H_1(t)\,\psi(a+it)|^2\,dt = O(U^{1/2}).$$

Thus (2.19) becomes

$$\int_T^{T+U} |\psi G(a+it)|^2\,dt = \int_T^{T+U} |\psi H(a+it)|^2\,dt + O(U^{1/2})$$
$$+ O(U^{1/4})\left(\int_T^{T+U} |\psi H(a+it)|^2\,dt\right)^{1/2}. \quad (2.20)$$

Most of the results that follow concern the evaluation of the first integral on the right of (2.20). By (2.17),

$$\int_T^{T+U} |\psi H(a+it)|^2\,dt = I_{11} + I_{22} + I_{33} + I_{12} + I_{21}$$
$$+ 2\,\mathrm{Re}\,I_{13} + 2\,\mathrm{Re}\,I_{23}\,, \quad (2.21)$$

where

$$I_{11} = \int_T^{T+U} |\psi g_1(a+it)|^2\,dt, \quad (2.22)$$

$$I_{22} = \int_T^{T+U} |\psi g_2(a+it)|^2\,dt/\log^2(t/2\pi), \quad (2.23)$$

$$I_{33} = \int_T^{T+U} |\chi_1|^2\,|\psi g_3(a+it)|^2\,dt/\log^2(t/2\pi), \quad (2.24)$$

$$I_{12} = -\int_T^{T+U} |\psi|^2\,g_1\bar{g}_2(a+it)\,dt/\log(t/2\pi) = \bar{I}_{21}\,, \quad (2.25)$$

$$I_{13} = \int_T^{T+U} |\psi|^2\,g_1\bar{\chi}_1\bar{g}_3(a+it)\,dt/\log(t/2\pi), \quad (2.26)$$

$$I_{23} = -\int_T^{T+U} |\psi|^2\,g_2\bar{\chi}_1\bar{g}_3(a+it)\,dt/\log^2(t/2\pi). \quad (2.27)$$

The evaluation of the I's which begins in §4 is similar in some respects to analysis done by Selberg [2].

3. Some Lemmas

Lemma 3.1. *Let $f(u)$ have a continuous derivative $f'(u)$ for $u \geqslant 1$. Let $f(u) \to 0$ as $u \to \infty$, and let $f(u)$ be monotone for $u \geqslant x$. Then for large x,*

$$\sum_{1 \leqslant n \leqslant x} f(n) = \int_1^x f(u)\,du + \tfrac{1}{2}f(1) - \int_1^\infty f'(u)([u] + 1/2 - u)\,du + O(f(x)), \tag{3.1}$$

where $[u]$ is the integral part of u.

Proof. The well-known proof follows.

$$\sum_{1 \leqslant n \leqslant x} f(n) = \int_1^x f(u)\,du + J,$$

where

$$\begin{aligned} J &= \int_{1-0}^x f(u)\,d([u] + 1/2 - u) \\ &= 1/2f(1) + ([x] + 1/2 - u)f(x) - \int_1^\infty f'(u)([u] + 1/2 - u)\,du \\ &\quad + \int_x^\infty f'(u)([u] + 1/2 - u)\,du. \end{aligned}$$

Since $f'(u)$ is of one sign for $u > x$ and $f(\infty) = 0$, the last integral is $O(f(x))$ and this proves the lemma.

Let A_1 and A_2 be positive integers, and let (A_1, A_2) denote the greatest common divisor of A_1 and A_2.

Lemma 3.2. *Let $1 \leqslant A_1, A_2 \leqslant T^{1/2}$. Let $(A_1, A_2) = 1$. Let $\sum$ denote a sum of over j_1, j_2 with $1 \leqslant j_1, j_2 \leqslant T^{1/2}$ and such that only the terms*

$$j_1 A_1 \neq j_2 A_2$$

are included. Then, with a satisfying (2.1) as usual,

$$\sum \frac{j_1^{-a} j_2^{-a}}{|\log(j_2 A_2 / j_1 A_1)|} \leqslant 100 T^{1-a} \log T. \tag{3.2}$$

Proof. Divide $\sum$ into two sums $\sum_1$ and $\sum_2$. For $\sum_1$, let

$$\frac{j_2 A_2}{j_1 A_1} \leqslant \frac{1}{2} \qquad \text{or} \qquad \frac{j_2 A_2}{j_1 A_1} \geqslant \frac{3}{2}.$$

Then

$$\sum\nolimits_1 \frac{j_1^{-a} j_2^{-a}}{|\log(j_2 A_2/j_1 A_1)|} < 4 \Big(\sum_{j_1 \leqslant T^{1/2}} j_1^{-a} \Big)^2$$

$$< 16 \left(\frac{T^{1/2-a/2}}{1-a} \right)^2 < 100 T^{1-a} \tag{3.3}$$

For $\sum_2$,

$$\frac{1}{2} \leqslant \frac{j_2 A_2}{j_1 A_1} \leqslant \frac{3}{2}$$

and so

$$\sum\nolimits_2 \frac{j_1^{-a} j_2^{-a}}{|\log(j_2 A_2/j_1 A_1)|} \leqslant 2 \sum\nolimits_2 \frac{j_1 A_1}{|j_2 A_2 - j_1 A_1|} \frac{1}{j_1^a j_2^a}$$

$$\leqslant 4 \sum \frac{(j_1 A_1 j_2 A_2)^{1/2}}{|j_2 A_2 - j_1 A_1|} \frac{1}{j_1^a j_2^a}$$

$$\leqslant 4 (A_1 A_2)^{1/2}\, T^{1/2-a} \sum \frac{1}{|j_2 A_2 - j_1 A_1|}. \tag{3.4}$$

The case $j_2 A_2 - j_1 A_1 \geqslant 1$ will be considered. The other case is the same if the order is reversed. Let $m \geqslant 1$. Let j_2' be the least $j_2 \geqslant 1$ for which there is a $j_1' \geqslant 1$ such that

$$j_2' A_2 - j_1' A_1 = m.$$

If

$$j_2 A_2 - j_1 A_1 = m, \tag{3.5}$$

then subtraction yields $(j_1 - j_2')A_2 = (j_1 - j_1')A_1$. Since $(A_1, A_2) = 1$, there is an n such that

$$j_2 = nA_1 + j_2', \qquad j_1 = nA_2 + j_1'. \tag{3.6}$$

The case $n < 0$ implies that either j_1 or $j_2 \leqslant 0$ by the definition of j_2' and so $n \geqslant 0$.

Let $A_M = \max\{A_1, A_2\}$. From (3.6) follows $nA_M \leqslant T^{1/2}$. So for fixed m the subset of j_1, j_2 for which (3.5) holds contains at most $2T^{1/2}/A_M$ elements. Thus

$$\sum\nolimits_2 \frac{1}{|j_2 A_2 - j_1 A_1|} \leqslant \frac{4T^{1/2}}{A_M} \sum_{m \leqslant 2T} \frac{1}{m}$$

$$< 5T^{1/2} L / A_M.$$

Used in (3.4) this gives

$$\sum\nolimits_2 \frac{j_1^{-a} j_2^{-a}}{|\log(j_2 A_2/jA_1)|} \leqslant 20T^{1-a}L.$$

With (3.3) this proves (3.2).

LEMMA 3.3. *There is a small $c > 0$ such that*

$$I = \int_{r(1-c)}^{r(1+c)} \exp\left[it \log \frac{t}{er}\right]\left(\frac{t}{2\pi}\right)^{1/2-a} dt$$
$$= (2\pi)^a r^{1-a} e^{-ir+\pi i/4} + O(r^{1/2-a})$$

for large r and a near $1/2$.

Proof. Let $t = r(1 + x)$ so that

$$I = (2\pi)^{a-1/2} r^{3/2-a} e^{-ir} I_1 ,$$

where

$$I_1 = \int_{-c}^{c} \exp[ir(1 + x)\log(1 + x) - irx](1 + x)^{1/2-a}\, dx.$$

Let

$$I_2 = \int_{-c}^{c} \exp[ir(1 + x)\log(1 + x) - irx]\, dx.$$

Then

$$I_1 - I_2 = \int_{-c}^{c} \exp[ir(1 + x)\log(1 + x) - irx] \log(1 + x)$$
$$\times \left\{\frac{\exp[(1/2 - a)\log(1 + x)] - 1}{\log(1 + x)}\right\} dx.$$

An integration by parts shows that

$$I_1 - I_2 = O(1/r).$$

Expanding $\log(1 + z)$ in a power series shows

$$(1 + z)\log(1 + z) - z = 1/2z^2[1 + zg(z)],$$

where $g(z)$ is analytic for $|z| < 1$. If

$$w = z[1 + zg(z)]^{1/2}$$

where the square root is 1 for $z = 0$, then there is a $c > 0$ such that for $|z| \leqslant c$

$$z = w + w^2 g_4(w),$$

where $g_4(w)$ is real for real w and is analytic. Let

$$-c_1 = -c[1 - cg(-c)]^{1/2}, \qquad c_2 = c[1 + cg(c)]^{1/2}.$$

Then if $g_5(w) = 2g_4 + wg_4{}'$,

$$I_2 = \int_{-c_1}^{c_2} e^{iru^2/2}(1 + ug_5(u))\, du$$

$$= \int_{-c_1}^{c_2} e^{iru^2/2}\, du + \int_{-c_1}^{c_2} e^{iru^2/2} g_5(u)\, d(u^2/2).$$

Integrating the second term by parts gives $O(1/r)$. Hence

$$I_2 = \int_{-\infty}^{\infty} e^{iru^2/2}\, du + J_1 + J_2 + O(1/r),$$

where

$$J_1 = \int_{c_2}^{\infty} e^{iru^2/2}\, du = \int_{c_2}^{\infty} e^{iru^2/2} d(u^2/2) u^{-1}$$

and J_2 is similar. Integration by parts shows $J_1 = O(1/r)$ and similarly for J_2. Hence

$$I_1 = \int_{-\infty}^{\infty} e^{iru^2/2}\, du + O(1/r).$$

But an elementary change of contour allows the evaluation of the integral to give

$$I_1 = (2\pi/r)^{1/2}\, e^{\pi i/4} + O(1/r),$$

which completes the proof.

The next lemma is similar to one in [2, Lemma 2].

LEMMA 3.4. *For large A and $A \leqslant r \leqslant B \leqslant A + A/\log A$,*

$$\int_A^B \exp[it \log(t/re)](t/2\pi)^{1/2-a}\, dt = (2\pi)^a\, r^{1-a} e^{-ir+\pi i/4} + E(r), \tag{3.7}$$

where a is such that $A^{1/2-a} = O(1)$ *and where*

$$E(r) = O(1) + O\left(\frac{A}{|A-r|+A^{1/2}}\right) + O\left(\frac{B}{|B-r|+B^{1/2}}\right). \quad (3.8)$$

For $r < A$ *or* $r > B$,

$$\int_A^B \exp[it \log(t/re)](t/2\pi)^{1/2-a}\, dt = E(r).$$

Proof. It will be convenient to use

$$F(t, r) = \exp[it \log(t/re)]. \quad (3.9)$$

Let $A + A^{1/2} \leqslant r \leqslant B - B^{1/2}$. Then

$$\int_A^B F(t, r)(t/2\pi)^{1/2-a}\, dt = \int_{r(1-c)}^{r(1+c)} F(t, r)(t/2\pi)^{1/2-a}\, dt + J_1 + J_2,$$

where

$$J_1 = \int_{r(1-c)}^{A} F(t, r)(t/2\pi)^{1/2-a}\, dt$$

and J_2 is the integral over $(B, r(1+c))$. Since $d/dt\, t \log(t/er) = \log t/r$, integration by parts gives

$$\begin{aligned}(2\pi)^{1/2-a} J_1 &= F(t, r)\, t^{1/2-a}/(\log t/r)i]_{r(1-c)}^{A} \\ &\quad + O(1) \int_{r(1-c)}^{A} dt/(t \log^2 r/t) \\ &\quad + O(1/\log r/A) \int_{r(1-c)}^{A} dt/t \\ &= O(1/\log r/A).\end{aligned}$$

For $A + A^{1/2} \leqslant r < 3/2\, A$,

$$\log \frac{r}{A} \geqslant \frac{1}{4} \frac{r-A}{A} \geqslant \frac{1}{8} \frac{r-A}{A} + \frac{1}{8A^{1/2}},$$

and so in any case $J_1 = E(r)$. Similarly $J_2 = E(r)$. By Lemma 3.3, (3.7) holds.

If $A - A^{1/2} < r < A + A^{1/2}$ then

$$\int_A^B F(t, r)\, t^{1/2-a}\, dt = O(A^{1/2}) + \int_{A+2A^{1/2}}^{B} F(t, r)\, t^{1/2-a}\, dt$$

(where the second term does not appear if $B \leqslant A + 2A^{1/2}$). Note that for the present range of r. $E(r)$ is of magnitude $A^{1/2}$. The integral on the right is integrated by parts to give

$$O\left(1\Big/\log\left(\frac{A+2A^{1/2}}{r}\right)\right) = O(A^{1/2}) = E(r),$$

and so again the lemma is valid. The case $B - B^{1/2} < r < B + B^{1/2}$ is treated similarly.

If $r < A - A^{1/2}$, one integration by parts establishes the lemma directly where r is considered first in the range $r \leqslant 3A/4$ and then $3A/4 < r < A - A^{1/2}$. The case $r > B + B^{1/2}$ is treated similarly.,

LEMMA 3.5. *For $m = 1, 2$, A large and $A \leqslant r \leqslant B < A + A/\log A$,*

$$\int_A^B \exp\left[it\log\frac{t}{er}\right]\left(\frac{t}{2\pi}\right)^{1/2-a}\frac{dt}{\log^m(t/2\pi)}$$
$$= (2\pi)^a r^{1-a}e^{-ir+\pi i/4}/\log^m(r/2\pi) + E(r)/\log^m A,$$

while for $r < A$ or $r > B$,

$$\int_A^B \exp\left[it\log\frac{t}{er}\right]\left(\frac{t}{2\pi}\right)^{1/2-a}\frac{dt}{\log^m(t/2\pi)} = \frac{E(r)}{\log^m A},$$

where $E(r)$ is (3.8).

Proof. The case $m = 1$ will be treated. The case $m = 2$ is similar. Using $F(t, r)$ as before, for $A - A^{1/2} < r < B + B^{1/2}$,

$$\int_A^B F(t,r)\left(\frac{t}{2\pi}\right)^{\frac{1}{2}-a}\frac{dt}{\log(t/2\pi)} = \frac{1}{\log(r/2\pi)}\int_A^B F(t,r)\left(\frac{t}{2\pi}\right)^{1/2-a}dt - J,$$

where

$$\begin{aligned}(2\pi)^{1/2-a}J &= \frac{1}{\log(r/2\pi)}\int_A^B F(t,r)\log\frac{t}{r}\,t^{1/2-a}\frac{dt}{\log(t/2\pi)}\\
&= \frac{1}{i\log(r/2\pi)}\left[\frac{t^{1/2-a}F(t,r)}{\log(t/2\pi)}\right]_A^B\\
&\quad + \frac{1}{i\log(r/2\pi)}\int_A^B\frac{F(t,r)\,t^{1/2-a}}{t\log(t/2\pi)}\left[\frac{1}{\log(t/2\pi)} - 1/2 + a\right]dt\\
&= O(\log^{-2}A).\end{aligned}$$

By Lemma 3.4, this proves the result in the above range of r. If $r < A - A^{1/2}$, then

$$\begin{aligned}
&\int_A^B F(t, r)\, t^{1/2-a}\, dt/\log(t/2\pi) \\
&\quad = \left[\frac{F(t,r)\, t^{1/2-a}}{i \log(t/2\pi) \log t/r}\right]_A^B \\
&\qquad + \frac{1}{i}\int_A^B \frac{F(t,r)\, t^{1/2-a}}{t \log(t/2\pi) \log t/r}\left[\frac{1}{\log t/r} + \frac{1}{\log t/2\pi} - 1/2 + a\right] dt \\
&\quad = O\left(\frac{1}{\log A \log A/r}\right) = \frac{E(r)}{\log A}.
\end{aligned}$$

The case $r > B + B^{1/2}$ is treated similarly.

LEMMA 3.6. *Let* $1 \leqslant k_1, k_2 \leqslant y$, *and let* $k = (k_1, k_2)$ *be the greatest common divisor. Then*

$$\sum \frac{k}{k_1 k_2} = O(\log^3 y).$$

Proof. Clearly,

$$\begin{aligned}
\sum \frac{k}{k_1 k_2} &\leqslant \sum \frac{1}{k_1 k_2} \sum_{j|k} j = \sum_{j \leqslant y} j \Big(\sum_{j|k_1} \frac{1}{k_1}\Big)^2 \\
&\leqslant \sum_{j \leqslant y} \frac{1}{j}\Big(\sum_{n \leqslant y/j} \frac{1}{n}\Big)^2 \leqslant \sum_{j \leqslant y} \frac{1}{j}\Big(\log \frac{y}{j} + 10\Big)^2 \\
&= O(\log^3 y).
\end{aligned}$$

LEMMA 3.7. *Let K be the region in the first quadrant given by*

$$C_1 \leqslant uv \leqslant C_2, \qquad C_3 u \leqslant v \leqslant C_4 u,$$

where $C_1 > 0$ *and* $C_3 > 0$. *Let* f, $\partial f/\partial u$ *and* $f/\partial v$ *be continous. Let* $|K|$ *be the area of K. Let* u_M *be the u-coordinate of the right-most point of K and* v_M *the v-coordinate of the highest point of K. Let n and m be integers. Then*

$$\sum_K f(n, m) = \iint_K f(u, v)\, du\, dv + J,$$

where $\sum$ *is over the lattice points in* K *and*

$$|J| \leqslant 2|f|_M(u_M + v_M + 1) + (|K| + 2v_M)\left|\frac{\partial f}{\partial v}\right|_M + |K|\left|\frac{\partial f}{\partial u}\right|_M,$$

where $|f|_M$ *is* max $|f|$ *in* K *and similarly for* $|\partial f/\partial u|_M$ *and* $|\partial f/\partial v|_M$.

Proof. Let the upper boundary of K be given by $v = g_M(u)$ and the lower boundary be $v = g_m(u)$, for $u_m \leqslant u \leqslant u_M$, where u_m is the u-coordinate of the left-most point of K and u_M of the right-most. Then

$$\sum_K f(n, m) = \sum \int_{g_m(n)}^{g_M(n)} f(n, v)\, dv - J_0, \tag{3.10}$$

where $\sum$ on the right is for $u_m \leqslant n \leqslant u_M$ and

$$J_0 = \sum \int_{g_m(n)}^{g_M(n)} f(n, v)\, d(v - [v]) = J_1 + J_2,$$

$$J_1 = \sum f(n, v)(v - [v])_{g_m(n)}^{g_M(n)},$$

$$J_2 = -\sum \int_{g_m(n)}^{g_M(n)} \frac{\partial f}{\partial v}(n, v)(v - [v])\, dv.$$

Clearly,

$$|J_1| \leqslant 2(u_M - u_m + 1)|f|_M$$

and

$$|J_2| \leqslant \sum (g_M(n) - g_m(n))\left|\frac{\partial f}{\partial v}\right|_M.$$

Integrating the term in $d(u - [u])$ by parts in what follows,

$$\sum (g_M(n) - g_m(n)) = \int_{u_m}^{u_M} (g_M(u) - g_m(u))\, du + J_3 + J_4$$

$$= |K| + J_3 + J_4,$$

where

$$J_3 = -(g_M(u) - g_m(u))(u - [u])]_{u_m}^{u_M} = O,$$

since $g_M(u_m) = g_m(u_m)$ and similarly at u_M and where

$$J_4 = \int_{u_m}^{u_M} (g_M'(u) - g_m'(u))(u - [u])\,du.$$

Thus

$$|J_4| \leqslant \int_{u_m}^{u_M} (|g_M'(u)| + |g_m'(u)|)\,du.$$

If v_L and v_R are the v-coordinates of the left-most and right-most points of K, then

$$\int_{u_m}^{u_M} |g_M'(u)|\,du = v_M - v_L + v_M - v_R\,,$$

$$\int_{u_m}^{u_M} |g_m'(u)|\,du = v_L - v_m + v_R - v_m\,.$$

Thus

$$|J_4| \leqslant 2(v_M - v_m)$$

and so

$$|J_2| \leqslant (|K| + 2v_M)\left|\frac{\partial f}{\partial v}\right|_M.$$

Proceeding similarly with the main term in (3.10),

$$\sum \int_{g_m(n)}^{g_M(n)} f(n, v)\,dv = \int_{u_m}^{u_M} du \int_{g_m(u)}^{g_M(u)} f(u, v)\,dv + J_5 + J_6 + J_7$$
$$= \iint_K f(u, v)\,du\,dv + J_5 + J_6 + J_7\,,$$

where

$$J_5 = -(u - [u]) \int_{g_m(u)}^{g_M(u)} f(u, v)\,dv]_{u_m}^{u_M} = O,$$

$$J_6 = \int_{u_m}^{u_M} (u - [u])\,du \int_{g_m(u)}^{g_M(u)} \frac{\partial f}{\partial u}(u, v)\,dv,$$

$$J_7 = \int_{u_m}^{u_M} (u - [u])\{f(u, g_M(u))\,g_M'(u) - f(u, g_m(u))\,g_m'(u)\}\,du.$$

Thus

$$|J_6| \leqslant |K| \left|\frac{\partial f}{\partial u}\right|_M,$$

and much as for J_4,

$$|J_7| \leqslant 2(v_M - v_m)\,|f|_M .$$

Taking account of J_1, J_2, J_5, J_6, and J_7, the lemma is proved.

LEMMA 3.8. *Let $f(n)$ be given. Let*

$$g(m) = \sum_{r|m} \mu(r) f(m/r).$$

Then

$$f(n) = \sum_{m|n} g(m)$$

The proof is easy and well-known.

LEMMA 3.9. *For large square-free j,*

$$\sum_{p|j} \frac{\log p}{p} = O(\log\log j),$$

where p is a prime number.

Proof. The function $\log x/x$ is decreasing for $x \geqslant 3$. Also $\log 2/2 > \log 5/5$. Let $q_1, q_2, q_3, \ldots$ be the sequence of increasing primes 2, 3, 5,... . Let r be chosen so that

$$q_1 q_2 \cdots q_r \leqslant j, \qquad q_1 q_2 \cdots q_r q_{r+1} > j.$$

Then for large j

$$\sum_{p|j} \frac{\log p}{p} \leqslant \sum_1^r \frac{\log q_j}{q_j}.$$

From a well-known elementary result,

$$\sum_1^r \frac{\log q_j}{q_j} = \log q_r + O(1).$$

Another elementary consequence of

$$\sum_1^r \log q_j \leqslant \log j$$

is $q_r \leqslant 2 \log j$. This proves the lemma.

LEMMA 3.10. *For large square-free j,*

$$\sum_{p|j} \frac{\log^2 p}{p} = O((\log\log j)^2).$$

Proof. The proof is very similar to that of Lemma 3.9. The function $\log^2 x/x$ is decreasing for $x > e^2$.

The next lemma is a special case of a result of [7].

LEMMA 3.11. *Let*

$$J(x) = \sum_{r \leq x} \frac{\mu^2(r) f(r)}{r},$$

where

$$f(r) = \prod_{p|r} f(p), \qquad f(p) = 1 + O(p^{-c}), \tag{3.11}$$

where $c > 0$. Then

$$J(x) = \prod \left(1 + \frac{f(p) - 1}{p+1}\right)\left(1 - \frac{1}{p^2}\right) \log x + O(1),$$

where $\prod$ is over all primes.

Proof. Let

$$A(p) = \frac{p(f(p)-1)}{p+1}, \qquad B(p) = \frac{p + f(p)}{p(f(p)-1)}.$$

Then

$$f(p) = A(p)(1 + B(p)).$$

Let

$$A(r) = \prod_{p|r} A(p), \qquad B(r) = \prod_{p|r} B(p).$$

Then

$$J(x) = \sum_{r \leq x} \frac{\mu^2(r)}{r} A(r) \sum_{m|r} B(m).$$

If $r = jm$, then

$$J(x) = \sum_{jm \leq x} \frac{\mu^2(jm)}{jm} A(jm)\, B(m)$$

$$= \sum_{m \leq x} \frac{\mu^2(m)}{m} A(m)\, B(m) \sum_{j \leq x/m}{}' \frac{\mu^2(j)}{j} A(j), \tag{3.12}$$

where $\sum'$ if for $(j, m) = 1$. By (3.11), there is a constant M_1 such that

$$| A(p)| \leqslant M_1/p^c,$$

and so if $w(r)$ is the number of distinct primes which are factors of r, then

$$| A(r)| \leqslant M_1^{w(r)}/r^c. \tag{3.13}$$

Obviously,

$$\sum_{j \leqslant x/m}' \frac{\mu^2(j)}{j} A(j) = \sum' \frac{\mu^2(j)}{j} A(j) + R_1 , \tag{3.14}$$

where $\sum'$ is over all $j \geqslant 1$, $(j, m) = 1$, and

$$| R_1 | \leqslant \sum_{j \geqslant x/m} \frac{\mu^2(j) M_1^{w(j)}}{j^{1+c}} .$$

It is a simple elementary fact that

$$w(r) \leqslant \frac{2 \log r}{\log \log r}$$

for large r. Choose j_1 large so the above holds for $r \geqslant j_1$ and also

$$\frac{2 \log M_1}{\log \log j} \leqslant \frac{c}{2}, \qquad j \geqslant j_1 .$$

Then

$$M_1^{w(j)} \leqslant j^{c/2}, \qquad j \geqslant j_1 .$$

Hence for some M_2 and all j,

$$M_1^{w(j)} \leqslant M_2 j^{c/2}. \tag{3.15}$$

Thus

$$| R_1 | \leqslant M_2 \sum_{j \geqslant x/m} j^{-1-c/2} \leqslant M_3 \left(\frac{m}{x}\right)^{c/2}$$

for some constant M_3. By writing the sum on the right in (3.14) as a product,

$$\sum_{j \leqslant x/m}' \frac{\mu^2(j)}{j} A(j) = \prod \left(1 + \frac{A(p)}{p}\right) \Big/ \prod_{p|m} \left(1 + \frac{A(p)}{p}\right) + R_1 .$$

Thus by (3.12),

$$J(x) = \prod \left(1 + \frac{A(p)}{p}\right) \sum_{m \leqslant x} \frac{\mu^2(m)}{m} \prod_{p|m} \frac{A(p)\,B(p)}{1 + A(p)/p} + R_2 ,$$

where

$$| R_2 | \leqslant M_3 x^{-c/2} \sum_{m \leqslant x} \frac{\mu^2(m)}{m^{1-c/2}} \, | A(m)\, B(m)|.$$

Since $A(p)\,B(p) = 1 + A(p)/p$,

$$J(x) = \prod \left(1 + \frac{A(p)}{p}\right) \sum_{m \leqslant x} \frac{\mu^2(m)}{m} + R_2$$

and

$$| A(m)\, B(m)| = \prod_{p|m} \left| 1 + \frac{A(p)}{p} \right| \leqslant \prod \left(1 + \frac{M_1}{p^{1+c}}\right) = M_4 , \quad (3.16)$$

where the last product is over all primes. Thus

$$| R_2 | \leqslant M_3 M_4 x^{-c/2} \sum_{m \leqslant x} m^{-1+c/2} = O(1),$$

and so

$$J(x) = \prod \left(1 + \frac{A(p)}{p}\right) \sum_{m \leqslant x} \frac{\mu^2(m)}{m} + O(1).$$

But

$$\sum_{m \leqslant x} \frac{\mu^2(m)}{m} = \sum_{m \leqslant x} \frac{1}{m} \sum_{j^2|m} \mu(j) = \sum_{rj^2 \leqslant x} \frac{\mu(j)}{rj^2}$$

$$= \sum_{r \leqslant x} \frac{1}{r} \sum_{j^2 \leqslant x/r} \frac{\mu(j)}{j^2} = \prod \left(1 - \frac{1}{p^2}\right) \sum_{r \leqslant x} \frac{1}{r} + R_3 ,$$

where

$$| R_3 | \leqslant \sum_{r \leqslant x} \frac{1}{r} \sum_{j^2 > x/r} \frac{1}{j^2} \leqslant \sum_{r \leqslant x} \frac{1}{r}\, O\left(\frac{r^{1/2}}{x^{1/2}}\right) = O(1).$$

Since

$$\sum_{r \leqslant x} \frac{1}{r} = \log x + O(1),$$

the lemma is proved.

LEMMA 3.12. *Let*

$$J(x) = \sum_{r \leqslant x} \frac{\mu^2(r) f(r)}{r} \log \frac{x}{r} ,$$

where f is as in Lemma 3.11. *Then*

$$J(x) = \frac{1}{2}\prod\left(1 + \frac{f(p) - 1}{p + 1}\right)\left(1 - \frac{1}{p^2}\right)\log^2 x + O(\log x).$$

Proof. In place of (3.12) now

$$J(x) = \sum_{m\leqslant x} \frac{\mu^2(m)\,A(m)\,B(m)}{m}\log\frac{x}{m}\sum_{j\leqslant x/m}{}' \frac{\mu^2(j)}{j}A(j) - R_4\,,$$

where

$$R_4 = \sum_{m\leqslant x} \frac{\mu^2(m)\,A(m)\,B(m)}{m}\sum_{j\leqslant x/m}{}' \frac{\mu^2(j)}{j}A(j)\log j.$$

From (3.13) and (3.15),

$$|A(j)| \leqslant M_2/j^{c/2},$$

and this with (3.16) leads easily to

$$R_4 = O(\log x).$$

The rest of the derivation is very similar to that in Lemma 3.11.

LEMMA 3.13. *Let*

$$J(x) = \sum_{r\leqslant x}\frac{\mu^2(r)f(r)}{r}\log^2\frac{x}{r}\,,$$

where f is as in Lemma 3.11. *Then*

$$J(x) = \frac{1}{3}\prod\left(1 + \frac{f(p) - 1}{p + 1}\right)\left(1 - \frac{1}{p^2}\right)\log^3 x + O(\log^2 x).$$

Proof. The procedure in Lemma 3.12 is elaborated in an obvious way

4. EVALUATION OF I_{11}

From (2.22),

$$I_{11} = \int_T^{T+U} \psi\bar{\psi}g_1\bar{g}_1(a + it)\,dt$$
$$= \int_T^{T+U} \sum \frac{b_{k_1}}{k_1^{a+it}}\frac{b_{k_2}}{k_2^{a-it}}\sum\frac{1}{j_1^{a+it}}\frac{1}{j_2^{a-it}}\,dt,$$

where the second $\sum$ is for $1 \leqslant j_1, j_2 \leqslant (t/2\pi)^{1/2}$. For k_1, k_2, $\sum$ is for $1 \leqslant k_1, k_2 \leqslant y$. Hence, $t \geqslant T$, $2\pi j_1^2 \leqslant t$, and $2\pi j_2^2 \leqslant t$. Thus, if

$$T_1 = \max(T, 2\pi j_1^2, 2\pi j_2^2),$$

then

$$I_{11} = \sum \frac{b_{k_1} b_{k_2}}{k_1{}^a k_2{}^a} \sum \frac{1}{j_1{}^a j_2{}^a} \int_{T_1}^{T+U} \exp\left(it \log \frac{j_2 k_2}{j_1 k_1}\right) dt, \tag{4.1}$$

and now $\sum$ for j_1, j_2 is for $1 \leqslant j_1 \leqslant \tau_1$ where $\tau_1 = ((T+U)/2\pi)^{1/2} < T^{1/2}$ and similarly $1 \leqslant j_2 \leqslant \tau_1$.

Let $k = (k_1, k_2)$ be the greatest common divisor of k_1 and k_2, and let $k_1 = kA_1$ and $k_2 = kA_2$. Then $(A_1, A_2) = 1$. Let

$$I_{11} = I_{11}' + I_{11}'', \tag{4.2}$$

where for I_{11}', $j_2 A_2 = j_1 A_1$, and for I_{11}'', $j_2 A_2 - j_1 A_1 \neq 0$. For I_{11}', $A_1 \mid j_2$ since $(A_1, A_2) = 1$. Thus, $j_2 = jA_1$ and hence $jA_1 A_2 = j_1 A_1$ so that $j_1 = jA_2$. Thus, from (4.1),

$$I_{11}' = \sum \frac{b_{k_1} b_{j_2}}{k_1{}^a k_2{}^a} \sum{}' \frac{1}{j^{2a} A_1{}^a A_2{}^a} (T + U - T_1),$$

where $\sum'$ is for $1 \leqslant j \leqslant \tau_1 / A_M$ where

$$A_M = \max\{A_1, A_2\}$$

and

$$T_1 = \max\{T, 2\pi j^2 A_M{}^2\}.$$

Since $k_1 = kA_1$, $k_2 = kA_2$,

$$I_{11}' = \sum \frac{b_{k_1} b_{k_2} k^{2a}}{k_1^{2a} k_2^{2a}} \sum{}' \frac{T + U - T_1}{j^{2a}}. \tag{4.3}$$

Clearly, since $| b_{k_1} | \leqslant 1$,

$$| I_{11}'' | \leqslant \sum \frac{1}{k_1{}^a k_2{}^a} \sum{}'' \frac{1}{j_1{}^a j_2{}^a} \frac{2}{| \log(j_2 k_2 / j_1 k_1) |},$$

where $\sum''$ is for $1 \leqslant j_1, j_2 \leqslant \tau_1$ and $j_2 A_2 \neq j_1 A_1$. By Lemma 3.2,

$$\begin{aligned} I_{11}'' &= O(T^{1-a} \log T) \sum \frac{1}{k_1{}^a k_2{}^a} \\ &= O(T^{1-a} y^{2-2a} \log T). \end{aligned}$$

Since $T^{1/2-a} = O(1)$,

$$I''_{11} = O(T^{1/2}yL) = O(T/L^{19}) = O(U/L^{9}). \tag{4.4}$$

Next,

$$I'_{11} = I_{11,1} + I_{11,2}, \tag{4.5}$$

where

$$I_{11,1} = U \sum \frac{b_{k_1} b_{k_2} k^{2a}}{k_1^{2a} k_2^{2a}} \sum \frac{1}{j^{2a}}. \tag{4.6}$$

$\sum$ is for $1 \leqslant j \leqslant \tau/A_M$, where $\tau = (T/2\pi)^{1/2}$, and

$$I_{11,2} = \sum \frac{b_{k_1} b_{k_2} k^{2a}}{k_1^{2a} k_2^{2a}} \sum{}'' \frac{T + U - 2\pi j^2 A_M{}^2}{j^{2a}},$$

and here $\sum''$ is for $\tau/A_M < j \leqslant \tau_1/A_M$. Hence, using (2.1),

$$I_{11,2} = O(1) \sum \frac{k}{k_1 k_2} \sum{}' \frac{U}{j^{2a}}.$$

But

$$\sum{}' \frac{1}{j^{2a}} \leqslant \left(\frac{A_M}{\tau}\right)^{2a} \left(\frac{\tau_1 - \tau}{A_M} + 1\right)$$
$$= O\left(\frac{\tau_1{}^2 - \tau^2}{\tau^{1+2a}}\right) + L^{-20} = O\left(\frac{U}{T}\right).$$

Hence,

$$I_{11,2} = O\left(\frac{U^2}{T}\right) \sum \frac{k}{k_1 k_2}.$$

By Lemma 3.6,

$$I_{11,2} = O\left(\frac{U^2 \log^3 T}{T}\right) = O\left(\frac{U}{L^7}\right). \tag{4.7}$$

Hence by (4.2), (4.4), (4.5), and (4.7),

$$I_{11} = I_{11,1} + O(U/L^7).$$

By Lemma 3.1 with C_1 a constant,

$$\sum_{1 \leqslant j \leqslant \tau/A_M} \frac{1}{j^{2a}} = \frac{1}{1-2a} \left(\frac{\tau}{A_M}\right)^{1-2a} - \frac{1}{1-2a} + C_1 + O\left(\frac{1}{L^{20}}\right),$$

where the last term follows from $A_M/\tau = 0(1/L^{20})$ since $A_M \leqslant y$. The above in $I_{11,1}$, (4.6), gives

$$I_{11} = \frac{\tau^{1-2a}U}{1-2a}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} - \left(\frac{1}{1-2a} - C_1\right)U\sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}} + O\left(\frac{U}{\log^7 T}\right), \tag{4.8}$$

where $k_m = \min\{k_1, k_2\}$, $k_M = \max\{k_1, k_2\}$.

5. The Evaluation of I_{22}

From (2.23),

$$I_{22} = \int_T^{T+U} \frac{\psi\bar{\psi}g_2\bar{g}_2\,dt}{\log^2 t/2\pi}$$
$$= \int_T^{T+U} \sum\frac{b_{k_1}b_{k_2}}{k_1^{a+it}k_2^{a-it}}\sum\frac{\log j_1 \log j_2}{j_1^{a+it}j_2^{a-it}}\frac{dt}{\log^2 t/2\pi}.$$

As in the case of I_{11}, summation and integration are inverted and the sum is separated into two parts: I'_{22} for $j_2A_2 = j_1A_1$ and I''_{22} for $j_2A_2 - j_1A_1 \neq 0$. Here in the case of I''_{22} there occurs

$$\int_{T_1}^{T+U} \exp\left(it\log\frac{j_2A_2}{j_1A_1}\right)\frac{dt}{\log^2 t/2\pi} = O\left(\frac{1}{L^2\,|\log(j_2A_2/j_1A_1)|}\right)$$

as shown by integration by parts. Since $\log j_1 < 2L$ and similarly for $\log j_2$, I''_{22} can be appraised by Lemma 3.2 just as was I''_{11} to give

$$I''_{22} = O(U/L^7). \tag{5.1}$$

For I'_{22}, in place of (4.3), the result now is

$$I'_{22} = \sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}}\sum{}'\frac{\log jA_1 \log jA_2}{j^{2a}}\int_{T_1}^{T+U}\frac{dt}{\log^2 t/2\pi}.$$

Since for $T_1 \leqslant t \leqslant T + U$,

$$\frac{1}{\log^2 t/2\pi} = \frac{1}{L^2} + O\left(\frac{U}{TL^3}\right),$$

$$I'_{22} = I_{22,1} + I_{22,2} + I_{22,3}, \tag{5.2}$$

where

$$I_{22,1} = \frac{U}{L^2}\sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}}\sum\frac{\log jA_1 \log jA_2}{j^{2a}}, \tag{5.3}$$

and as for $I_{11,1}$, the inner sum is for $1 \leqslant j \leqslant \tau/A_M$. Here

$$I_{22,2} = \frac{1}{L^2}\sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}}\sum{}''\frac{\log jA_1 \log jA_2}{j^{2a}}(T + U - 2\pi j^2 A_M{}^2),$$

and as for $I_{11,2}$, $\sum''$ is for $\tau/A_M < j \leqslant \tau_1/A_M$. Finally,

$$I_{22,3} = O\left(\frac{U^2}{TL^3}\right) \sum \frac{k^{2a}}{k_1^{2a} k_2^{2a}} \sum_{1 \leqslant j \leqslant \tau_1/A_M} \frac{\log jA_1 \log jA_2}{j^{2a}}.$$

Because $T^{1-2a} = O(1)$, the inner sum is $O(L^3)$. Thus

$$I_{22,3} = O\left(\frac{U^2}{T}\right) \sum \frac{k}{k_1 k_2},$$

and by Lemma 3.6,

$$I_{22,3} = O\left(\frac{U}{L^7}\right).$$

As regards $I_{22,2}$, treating it much like $I_{11,2}$ gives

$$I_{22,2} = O\left(\frac{U}{L^7}\right).$$

Hence

$$I_{22} = I_{22,1} + O\left(\frac{U}{L^7}\right).$$

The inner sum in $I_{22,1}$, (5.3), is evaluated by Lemma 3.1 to give

$$\sum \frac{\log jA_1 \log jA_2}{j^{2a}} = \int_1^{\tau/A_M} \frac{\log vA_1 \log vA_2}{v^{2a}}\, dv + C_2 \log A_1 \log A_2 + C_3 \log A_1 + C_4 + O(A_M L^2/\tau),$$

where the term in $\log A_2$ is replaced by one in $\log A_1$ which is allowed by symmetry of the outer sum in (5.3). The term in $O(A_M/\tau)$ is $O(L^{-20})$ as in I_{11} and is incorporated into the error term $O(U/L^7)$. Integrating by parts gives

$$\begin{aligned}
\int_1^{\tau/A_M} \frac{\log vA_1 \log vA_2}{v^{2a}}\, dv &= \frac{v^{1-2a} \log vA_1 \log vA_2}{(1-2a)} \\
&\quad - \frac{v^{1-2a}(\log vA_1 + \log vA_2)}{(1-2a)^2} + 2\frac{v^{1-2a}}{(1-2a)^3}\Big]_1^{\tau/A_M} \\
&= \frac{\tau^{1-2a} k^{1-2a}}{(1-2a)\, k_M^{1-2a}} \log \tau \log \frac{\tau k_m}{k_M} \\
&\quad - \frac{\tau^{1-2a} k^{1-2a}}{(1-2a)^2\, k_M^{1-2a}} \left(\log \tau + \log \frac{\tau k_m}{k_M}\right) + \frac{2\tau^{1-2a} k^{1-2a}}{(1-2a)^3\, k_M^{1-2a}} \\
&\quad - \frac{1}{1-2a} \log \frac{k_1}{k} \log \frac{k_2}{k} + \frac{1}{(1-2a)^2}\left(\log \frac{k_1}{k} + \log \frac{k_2}{k}\right) \\
&\quad - \frac{2}{(1-2a)^3}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
I_{22} = {} & \frac{U\tau^{1-2a}}{2L(1-2a)}\sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} \log \frac{\tau k_m}{k_M} \\
& - \frac{U\tau^{1-2a}}{L^2(1-2a)^2}\sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} \left(\tfrac{1}{2}L + \log \frac{\tau k_m}{k_M}\right) \\
& + \frac{2U\tau^{1-2a}}{L^2(1-2a)^3}\sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} \\
& - \frac{U}{L^2}\left(\frac{1}{1-2a} - C_2\right)\sum \frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}} \log \frac{k_1}{k} \log \frac{k_2}{k} \\
& + \frac{U}{L^2}\left(\frac{2}{(1-2a)^2} + C_3\right)\sum \frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}} \log \frac{k_1}{k} \\
& - \frac{U}{L^2}\left(\frac{2}{(1-2a)^3} - C_4\right)\sum \frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}} + O(U/L^7).
\end{aligned}
$$

6. The Evaluation of I_{33}

From (2.24),

$$
\begin{aligned}
I_{33} &= \int_T^{T+U} \psi\bar{\psi} g_3 \bar{g}_3 \left(\frac{t}{2\pi}\right)^{1-2a} \frac{dt}{\log^2 t/2\pi} \\
&= \int_T^{T+U} \sum \frac{b_{k_1}b_{k_2}}{k_1^{a+it}k_2^{a-it}} \sum \frac{\log j_1 \log i_2}{j_1^{1-a+it}j_2^{1-a-it}} \left(\frac{t}{2\pi}\right)^{1-2a} \frac{dt}{\log^2 t/2\pi}.
\end{aligned}
$$

As in the case of I_{11}, the summation and integration are inverted and the sum separated into two parts I'_{33} for $j_2A_2 = j_1A_1$ and I''_{33} for $j_2A_2 - j_1A_1 \neq 0$. Integration by parts shows as before that

$$
\int_{T_1}^{T+U} \exp\left(it \log \frac{j_2A_2}{j_1A_1}\right)\left(\frac{t}{2\pi}\right)^{1-2a} \frac{dt}{\log^2 t/2\pi} = O\left(\frac{1}{L^2 \,|\log(j_1A_2/jA_1)|}\right),
$$

and much as for I''_{22},

$$
I''_{33} = O(U/\log^7 T).
$$

For I'_{33}, the result is

$$
I'_{33} = \sum \frac{b_{k_1}b_{k_2}k^{2-2a}}{k_1k_2} {\sum}' \frac{\log jA_1 \log jA_2}{j^{2-2a}} \int_{T_1}^{T+U} \left(\frac{t}{2\pi}\right)^{1-2a} \frac{dt}{\log^2 t/2\pi},
$$

where the inner sum if for $1 \leqslant j \leqslant \tau_1/A_M$.

By the mean value theorem,

$$\left(\frac{t}{2\pi}\right)^{1-2a}\frac{1}{\log^2 t/2\pi}-\left(\frac{T}{2\pi}\right)^{1-2a}\frac{1}{L^2}=(t-T)\,O\left(\frac{1}{TL^3}\right)=O\left(\frac{U}{TL^3}\right). \quad (6.1)$$

Now

$$I'_{33}=I_{33,1}+I_{33,2}+I_{33,3}$$

much as for I'_{22} and $I_{33,2}$ and $I_{33,3}$ are disposed of as before. Hence with

$$I_{33,1}=\frac{\tau^{2-4a}U}{L^2}\sum\frac{b_{k_1}b_{k_2}k^{2-2a}}{k_1k_2}\sum\frac{\log jA_1\log jA_2}{j^{2-2a}},$$

where the inner sum is for $1\leqslant j\leqslant \tau/A_M$,

$$I_{33}=I_{33,1}+O(U/\log^7 T).$$

The inner sum in $I_{33,1}$ is evaluated by Lemma 3.1 to give

$$\sum\frac{\log jA_1\log jA_2}{j^{2-2a}}=\int_1^{\tau/A_M}\frac{\log vA_1\log vA_2}{v^{2-2a}}\,dv$$
$$+C_5\log A_1\log A_2+C_6\log A_1+C_7$$
$$+O(A_ML^2/\tau).$$

The last term is treated much as for $I_{11,1}$. Integration by parts gives

$$\int_1^{\tau/A_M}\frac{\log vA_1\log vA_2}{v^{2-2a}}\,dv$$
$$=\frac{v^{-1+2a}}{-(1-2a)}\log vA_1\log vA_2$$
$$-\frac{v^{-1+2a}}{(1-2a)^2}(\log vA_1+\log vA_2)-\frac{2v^{-1+2a}}{(1-2a)^3}\Big]_1^{\tau/A_M}$$
$$=-\frac{\tau^{-1+2a}k_M^{1-2a}}{(1-2a)\,k^{1-2a}}\log\tau\log\frac{\tau k_m}{k_M}$$
$$-\frac{\tau^{-1+2a}k_M^{1-2a}}{(1-2a)^2\,k^{1-2a}}\left(\log\tau+\log\frac{\tau k_m}{k_M}\right)-\frac{2\tau^{-1+2a}k_M^{1-2a}}{(1-2a)^3\,k^{1-2a}}$$
$$+\frac{1}{1-2a}\log\frac{k_1}{k}\log\frac{k_2}{k}+\frac{1}{(1-2a)^2}\left(\log\frac{k_1}{k}+\log\frac{k_2}{k}\right)$$
$$+\frac{2}{(1-2a)^3}.$$

Hence,

$$I_{33} = -\frac{\tau^{1-2a}U}{2L(1-2a)}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\log\frac{\tau k_m}{k_M}$$
$$-\frac{\tau^{1-2a}U}{(1-2a)^2L^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\left(\tfrac{1}{2}L+\log\frac{\tau k_m}{k_M}\right)$$
$$-\frac{2\tau^{1-2a}U}{(1-2a)^3L^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}$$
$$+\frac{\tau^{2-4a}U}{L^2}\left(\frac{1}{1-2a}+C_5\right)\sum\frac{b_{k_1}b_{k_2}k^{2-2a}}{k_1k_2}\log\frac{k_1}{k}\log\frac{k_2}{k}$$
$$+\frac{\tau^{2-4a}U}{L^2}\left(\frac{2}{(1-2a)^2}+C_6\right)\sum\frac{b_{k_1}b_{k_2}k^{2-2a}}{k_1k_2}\log\frac{k_1}{k}$$
$$+\frac{\tau^{2-4a}U}{L^2}\left(\frac{2}{(1-2a)^3}+C_7\right)\sum\frac{b_{k_1}b_{k_2}k^{2-2a}}{k_1k_2}+O\left(\frac{U}{L^7}\right). \tag{6.2}$$

7. Evaluation of I_{12}

From (2.25),

$$I_{12} = -\int_T^{T+U}\psi\bar{\psi}g_1\bar{g}_2\frac{dt}{\log t/2\pi},$$
$$I_{21} = -\int_T^{T+U}\psi\bar{\psi}\bar{g}_1g_2\frac{dt}{\log t/2\pi},$$
$$I_{12} = -\int_T^{T+U}\sum\frac{b_{k_1}b_{k_2}}{k_1^{a+it}k_2^{a-it}}\sum\frac{1}{j_1^{a+it}}\frac{\log j_2}{j_2^{a-it}}\frac{dt}{\log t/2\pi}.$$

As in the case of I_{11}, the order of summation and integration are inverted and the sum is separated into I'_{12} for $j_2A_2 = j_1A_1$ and I''_{12} for $j_2A_2 - j_1A_1 \neq 0$. The case of I''_{12} leads as for I''_{11} to

$$I''_{12} = O(U/L^7).$$

Just as for I'_{22}, here I'_{12} finally yields

$$I_{12} = I_{12,1} + O(U/L^7).$$

Moreover,

$$I_{12} + I_{21} = I_{12,1} + I_{21,1} + O(U/\log^7 T),$$

where

$$I_{12,1} + I_{21,1} = -\frac{U}{L}\sum \frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}} \sum \frac{\log jA_1 + \log jA_2}{j^{2a}},$$

and the inner sum is for $1 \leqslant j \leqslant \tau/A_M$. By Lemma 3.1,

$$\sum \frac{\log jA_1 + \log jA_2}{j^{2a}} = \int_1^{\tau/A_M} \frac{\log vA_1 + \log vA_2}{v^{2a}}\,dv + C_8 \log A_1 + C_9 + O(A_M L/\tau),$$

where C_8 includes also the coefficient of $\log A_2$ which is legitimate because of symmetry. Integration by parts shows

$$\begin{aligned}
&\int_1^{\tau/A_M} \frac{\log vA_1 + \log vA_2}{v^{2a}}\,dv \\
&= \frac{v^{1-2a}}{1-2a}(\log vA_1 + \log vA_2) - \frac{2v^{1-2a}}{(1-2a)^2}\Big]_1^{\tau/A_M} \\
&= \frac{\tau^{1-2a}k^{1-2a}}{(1-2a)\,k_M^{1-2a}}\left(\log\tau + \log\frac{\tau k_m}{k_M}\right) - \frac{2\tau^{1-2a}k^{1-2a}}{(1-2a)^2\,k_M^{1-2a}} \\
&\quad - \frac{1}{1-2a}\left(\log\frac{k_1}{k} + \log\frac{k_2}{k}\right) + \frac{2}{(1-2a)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
2I_{12} = I_{12} + I_{21} = &-\frac{U\tau^{1-2a}}{(1-2a)L}\sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\left(\tfrac{1}{2}L + \log\frac{\tau k_m}{k_M}\right) \\
&+ \frac{2U\tau^{1-2a}}{(1-2a)^2 L}\sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} \\
&+ \frac{U}{L}\left(\frac{2}{1-2a} - C_8\right)\sum \frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}}\log\frac{k_1}{k} \\
&- \frac{U}{L}\left(\frac{2}{(1-2a)^2} + C_9\right)\sum \frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}} \\
&+ O(U/L^7).
\end{aligned} \tag{7.1}$$

8. The Evaluation of I_{13}

From (2.26),

$$I_{13} = \int_T^{T+U} \psi\bar{\psi} g_1 \bar{g}_3 \bar{\chi}_1\,dt/\log t/2\pi.$$

The computation here is different from the previous ones and more complicated. Here,

$$I_{13} = e^{-\pi i/4} \sum \frac{b_{k_1} b_{k_2}}{k_1{}^a k_2{}^a} \sum \frac{1}{j_1{}^a} \frac{\log j_2}{j_2^{1-a}} \int_{T_1}^{T+U} \left(\frac{t}{2\pi}\right)^{1/2-a}$$

$$\times \exp\left[it \log \frac{tk_2}{2\pi e j_1 j_2 k_1}\right] \frac{dt}{\log t/2\pi},$$

where as before $1 \leqslant k_1, k_2 \leqslant y$ and $1 \leqslant j_1, j_2 \leqslant \tau_1 < T^{1/2}$. Using Lemma 3.5,

$$I_{13} = I'_{13} + I''_{13},$$

where

$$|I''_{13}| \leqslant \frac{1}{L} \sum \frac{1}{k_1{}^a k_2{}^a} \sum \frac{\log j_2}{j_1{}^a j_2^{1-a}} E\left(\frac{2\pi j_1 j_2 k_1}{k_2}\right) \tag{8.2}$$

with the above limits and where

$$I'_{13} = 2\pi \sum \frac{b_{k_1} b_{k_2} k_1^{1-2a}}{k_2} \sum \frac{j_1^{1-2a} \log j_2}{\log(j_1 j_2 k_1/k_2)} \exp[-2\pi i j_1 j_2 k_1/k_2], \tag{8.3}$$

where for I'_{13} the inner sum is for

$$T_1 \leqslant 2\pi j_1 j_2 k_1/k_2 \leqslant T + U$$

or, since $T_1 = \max\{T, 2\pi j_1{}^2, 2\pi j_2{}^2\}$, for

$$\begin{aligned} &\frac{Tk_2}{2\pi k_1} \leqslant j_1 j_2 \leqslant \frac{(T+U)\,k_2}{2\pi k_1}; \\ &\frac{j_1 k_2}{k_1} \leqslant j_2 \leqslant \frac{j_1 k_1}{k_2}. \end{aligned} \tag{8.4}$$

To appraise I''_{13}, note that if $2\pi j_1 j_2 k_1/k_2 \leqslant 3T/4$ or $2\pi j_1 i_2 k_1/k_2 \geqslant 5T/4$, then $E = O(1)$, and so this part of I''_{13} is dominated by

$$O(1/L) \sum (\log j_2)/(k_1 k_2 j_1 j_2)^{1/2} = O(yT) = O(U/L^{10}) \tag{8.5}$$

so that only the case

$$3T/4 < 2\pi j_1 j_2 k_1/k_2 < 5T/4 \tag{8.6}$$

remains. The term $O(1)$ is E is already taken care of above. The next term in E is associated with

$$T/(|\, T_1 - 2\pi j_1 j_2 k_1/k_2 \,| + T^{1/2}).$$

Using (8.6) in (8.2) shows that the part of I''_{13} associated with the above term is dominated by a constant times I'''_{13}, where

$$I'''_{13} = \frac{1}{T^{1/2}} \sum \frac{1}{k_2} \sum \frac{T}{| T_1 - 2\pi j_1 j_2 k_1/k_2 | + T^{1/2}}.$$

This will be summed over all $j_1, j_2 \leqslant \tau_1$ and $k_1, k_2 \leqslant y$. There are three cases. First, for $j_2 \leqslant \tau$ and $j_1 \leqslant \tau$, $T_1 = T$. This part of I'''_{13} is given by

$$\frac{1}{T^{1/2}} \sum \frac{1}{k_2} \sum \frac{Tk_2/(2\pi j_2 k_1)}{| T_1 k_2/(2\pi j_2 k_1) - j_1 | + T^{1/2} k_2/(2\pi j_2 k_1)}. \tag{8.7}$$

Summing first on j_1, if the absolute value term in the denominator of the innermost expression is less than 1/2, then the quotient is at most $T^{1/2}$. Discarding the last term in the denominator, the rest of the sum on j_1 is dominated by

$$(2L + O(1))\, Tk_2/(2\pi j_2 k_1).$$

Hence, this part of I'''_{13} is dominated by two sums. The first is

$$O(1) \sum_{k_1, k_2} \frac{1}{k_2} \sum_{j_2} 1 = O(T^{1/2} y L) = O(U/L^9).$$

The second sum is

$$T^{1/2}(2L + O(1)) \sum_{k_1, k_2} \frac{1}{k_1} \sum_{j_2} \frac{1}{j_2} = O(T^{1/2} y L^3) = O(U/L^7).$$

If, for $\tau < j_2 \leqslant \tau_1$, the sum in the right term of (8.7) is for $j_1 \leqslant j_2$, then $T_1 = 2\pi j_2^2$. The procedure is still as above. Finally, for $\tau < j_1 \leqslant \tau_1$ and $j_1 > j_2$, the sum is carried out first on j_2 with $T_1 = 2\pi j_1^2$ and with obvious changes in the details. Hence, in all cases,

$$I'''_{13} = O(U/L^7). \tag{8.8}$$

For the last term of E, B is $T + U$, and so unlike the above, there is now only one case with the first sum always on j_1. The result is as for I'''_{13}. All this yields

$$I''_{13} = O(U/\log^7 T)$$

and so

$$I_{13} = I'_{13} + O(U/\log^7 T). \tag{8.9}$$

Turning next to I'_{13}, (8.3), let $k_1 = kA_1$ and $k_2 = kA_2$ as before. Suppose first that j_2 is restricted so that

$$A_1 j_2 \equiv j (\mathrm{mod}\, A_2), \qquad 1 \leqslant j \leqslant A_2 - 1,$$

and denote this part of the sum of I'_{13} by $I'_{13,1}$. Clearly,

$$\sum_{j_1} \exp[-2\pi i j_1 j_2 A_1/A_2] = O\left(\frac{A_2}{j} + \frac{A_2}{A_2 - j}\right).$$

Hence, using partial summation and recalling (8.4),

$$\sum_{j_1} \frac{j_1^{1-2a}}{\log(j_1 j_2 k_1/k_2)} \exp[-2\pi i j_1 j_2 A_1/A_2] = O\left(\frac{A_2}{j} + \frac{A_2}{A_2 - j}\right).$$

Since $(A_1, A_2) = 1$, whenever j_2 goes through any successive set of integers of length A_2, $A_1 j_2$ goes through the whole residue set (mod A_2).

Summing next in j_2, j_2 goes through at most τ_1/A_2 successive sets of integers each of length $A_2 - 1$ and for each set j goes from 1 to $A_2 - 1$. Hence,

$$\sum_{j_2, j_1} \frac{j_1^{1-2a} \log j_2}{\log j_1 j_2 k_1/k_2} \exp[-2\pi i j_1 j_2 A_1/A_2]$$

$$= O\left(\frac{\tau_1 L}{A_2} + L\right) \sum_{j=1}^{A_2 - 1} \left(\frac{A_2}{j} + \frac{A_2}{A_2 - j}\right) = (OT^{1/2}L^2).$$

Thus,

$$I'_{13,1} = O(T^{1/2}L^2) \sum_{k_1, k_2} \frac{1}{k_2} = O(yT^{1/2}L^3) = O(U/L^7).$$

Denote the part of I'_{13} for which $A_2 \mid j_2$ by $I_{13,1}$. Then from (8.9) and the above,

$$I_{13} = I_{13,1} + O(U/L^7).$$

Let $j_2 = jA_2$. Then by (8.3)

$$I_{13,1} = 2\pi \sum \frac{b_{k_1} b_{k_2}}{k_2} k_1^{1-2a} \sum_K j_1^{1-2a} \frac{\log jA_2}{\log j_1 j A_1}, \tag{8.10}$$

where K will be described. By (8.4), the inner sum is for

$$\frac{T_1}{2\pi A_1} \leqslant j_1 j \leqslant \frac{T + U}{2\pi A_1}; \qquad \frac{j_1}{A_1} \leqslant j \leqslant \frac{j_1 A_1}{A_2^2}. \tag{8.11}$$

Thus, K denotes the region in the j_1, j plane which satisfies the above inequalities. Note that K is empty unless $k_2 \leqslant k_1$. An easy computation shows that in K

$$\tau/A_1 \leqslant j \leqslant \tau_1/A_2, \tag{8.12}$$

since τ/A_1 and τ_1/A_2 are the least and greatest height of K, respectively. Summing first on j_1 in K,

$$\begin{aligned} \sum_{j,j_1 \in K} 1 &\leqslant \sum_j \left(\frac{U}{2\pi A_1 j} + 1\right) \\ &\leqslant \frac{UL}{\pi A_1} + \frac{T^{1/2}}{A_2}, \end{aligned} \tag{8.13}$$

where use is made of (8.12).

Let

$$I_{13,2} = \frac{2\pi}{L} \sum \frac{b_{k_1} b_{k_2}}{k_2} k_1^{1-2a} \sum_K j_1^{1-2a} \log jA_2. \tag{8.14}$$

Due to (8.11),

$$\frac{1}{\log j_1 j A_1} - \frac{1}{L} = O\left(\frac{\log(1 + U/T)}{L^2}\right) = O\left(\frac{1}{L^{12}}\right).$$

Hence,

$$I_{13,1} = I_{13,2} + O\left(\frac{1}{L^{12}}\right) \sum_{k_1,k_2} \frac{1}{k_2} \sum_K L.$$

By (8.13),

$$I_{13,1} = I_{13,2} + O\left(\frac{U}{L^{10}}\right) \sum \frac{k}{k_1 k_2} + O\left(\frac{T^{1/2}}{L^{11}}\right) \sum_{k_1,k_2} \frac{1}{k_2}.$$

By Lemma 3.6,

$$\begin{aligned} I_{13,1} &= I_{13,2} + O\left(\frac{U}{L^7}\right) + O\left(\frac{T^{1/2} y}{L^{10}}\right) \\ &= I_{13,2} + O\left(\frac{U}{L^7}\right). \end{aligned} \tag{8.15}$$

Thus,

$$I_{13} = I_{13,2} + O\left(\frac{U}{L^7}\right). \tag{8.16}$$

Now, Lemma 3.7 will be used for the inner sum of $I_{13,2}$. In the notation of Lemma 3.7, $|K|$ satisfies

$$|K| \leqslant \frac{UL}{\pi A_1}$$

by proceeding much as in the derivation of (8.13).

It has already been seen that $v_m = \tau/A_1$ and $v_M = \tau_1/A_2$. An easy computation shows that $u_M = \tau_1$ and $u_m = \tau A_2/A_1$. For the interior sum in (8.14),

$$f(u, v) = u^{1-2a} \log vA_2$$

and so

$$|f|_M = O(L), \qquad \left|\frac{\partial f}{\partial u}\right|_M = O\left(\frac{L}{u}\right), \qquad \left|\frac{\partial f}{\partial v}\right|_M = O\left(\frac{1}{v}\right).$$

Replacing u by u_m and v by v_m,

$$|f|_M = O(L), \qquad \left|\frac{\partial f}{\partial u}\right|_M = O\left(\frac{LA_1}{\tau A_2}\right), \qquad \left|\frac{\partial f}{\partial v}\right| = O\left(\frac{A_1}{\tau}\right).$$

Thus by Lemma 3.7,

$$J = O\left(L\tau_1 + \frac{UL}{\tau} + \frac{A_1}{A_2} + \frac{UL^2}{\tau A_2}\right) = O(LT^{1/2})$$

and

$$\sum_K j_1^{1-2a} \log jA_2 = F + O(LT^{1/2}), \tag{8.17}$$

where

$$F = \iint_K u^{1-2a} \log vA_2 \, du \, dv.$$

Using polar coordinates,

$$F = \iint_K r^{1-2a}(\cos\theta)^{1-2a} \log(r \sin\theta A_2) r \, dr \, d\theta, \tag{8.18}$$

where now K takes the form

$$\frac{T}{2\pi A_1 \sin\theta\cos\theta} \leqslant r^2 \leqslant \frac{T+U}{2\pi A_1 \sin\theta\cos\theta}, \qquad \frac{1}{A_1} \leqslant \tan\theta \leqslant \frac{A_1}{A_2{}^2}.$$

Let $r^2 = x$ so that

$$F = \frac{1}{4} \int_{\theta_1}^{\theta_2} (\cos\theta)^{1-2a}\, d\theta \int_{x_1}^{x_2} x^{1/2-a} \log(x \sin^2\theta\, A_2^2)\, dx, \tag{8.19}$$

where

$$x_1 = \frac{T}{2\pi A_1 \sin\theta\cos\theta}, \qquad x_2 = \frac{T+U}{2\pi A_1 \sin\theta \cos\theta}$$

$$\theta_1 = \tan^{-1}(1/A_1), \qquad \theta_2 = \tan^{-1}(A_1/A_2^2).$$

For an $f(x)$ of class C_1, Taylor's theorem with a remainder gives

$$\int_{x_1}^{x_2} f(x)\, dx = (x_2 - x_1) f(x_1) + \tfrac{1}{2}(x_2 - x_1)^2\, O(|f'|_M),$$

where $|f'|_M$ is max $|f'|$ on $x_1 \leqslant x \leqslant x_2$. From this,

$$\int_{x_1}^{x_2} x^{1/2-a} \log(x \sin^2\theta\, A_2^2)\, dx$$
$$= \left(\frac{T}{2\pi}\right)^{1/2-a} \frac{U}{2\pi(A_1 \sin\theta\cos\theta)^{(3/2)-a}} \log\left(\frac{TA_2^2 \tan\theta}{2\pi A_1}\right)$$
$$+ O\left(\frac{U^2 L}{TA_1 \sin\theta\cos\theta}\right).$$

Hence,

$$F = F_1 + F_2,$$

where

$$F_1 = \frac{T^{1-2a} U}{8\pi A_1^{(3/2)-a}} \int_{\theta_1}^{\theta_2} \frac{(\cot\theta)^{1/2-a}}{\sin\theta\cos\theta} \log\left(\frac{TA_2^2\tan\theta}{2\pi A_1}\right) d\theta$$

and

$$F_2 = O\left(\frac{U^2 L}{TA_1}\right) \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin\theta\cos\theta}.$$

If the change of variables $\tan\theta = v$ is made, then

$$F_1 = \frac{T^{1-2a} U}{8\pi A_1^{(3/2)-a}} \int_{1/A_1}^{A_1/A_2^2} \log\left(\frac{TA_2^2 v}{2\pi A_1}\right) \frac{dv}{v^{(3/2)-a}} \tag{8.20}$$

and

$$F_2 = O\left(\frac{U^2L}{TA_1}\right) \int_{1/A_1}^{A_1/A_2^2} \frac{dv}{v} = O\left(\frac{U^2L^2}{TA_1}\right) = O\left(\frac{U}{A_1 L^8}\right).$$

Hence,

$$F = F_1 + O\left(\frac{U}{A_1 L^8}\right). \tag{8.21}$$

Using this, (8.14), (8.17), and Lemma 3.6,

$$\begin{aligned} I_{13,2} &= I_{13,3} + O(LT^{1/2}y) + O(U/L^5) \\ &= I_{13,3} + O(U/L^5), \end{aligned} \tag{8.22}$$

where

$$I_{13,3} = \frac{2\pi}{L} \sum{}' \frac{b_{k_1} b_{k_2}}{k_2} k_1^{1-2a} F_1 . \tag{8.23}$$

Here, $\sum'$ denotes $k_2 \leqslant k_1$ since K is empty otherwise. From (8.16) and (8.22),

$$I_{13} = I_{13,3} + O(U/L^5). \tag{8.24}$$

F_1, (8.20), is evaluated by integration by parts to give

$$\begin{aligned} F_1 = &- \frac{\tau^{1-2a} U}{8\pi A_1^{(3/2)-a}} \frac{v^{-1/2+a}}{\frac{1}{2} - a} \log\left(\frac{TA_2{}^2 v}{2\pi A_1}\right) \Big|_{1/A_1}^{A_1/A_2{}^2} \\ &- \frac{\tau^{1-2a} U}{8\pi A_1^{(3/2)-a}} \frac{v^{-1/2+a}}{(\frac{1}{2} - a)^2} \Big|_{1/A_1}^{A_1/A_2{}^2} \\ = &- \frac{\tau^{1-2a} U}{4\pi(1-2a)} \frac{A_2^{1-2a}}{A_1^{2-2a}} L \\ &+ \frac{\tau^{1-2a} U}{2\pi(1-2a) A_1} \log \frac{\tau k_2}{k_1} \\ &- \frac{\tau^{1-2a} U}{2\pi(1-2a)^2} \frac{A_2^{1-2a}}{A_1^{2-2a}} + \frac{\tau^{1-2a} U}{2\pi(1-2a)^2} \frac{1}{A_1} . \end{aligned} \tag{8.25}$$

Used in (8.23) and with (8.24),

$$\begin{aligned} I_{13} = &- \frac{\tau^{1-2a} U}{2(1-2a)} \sum{}' \frac{b_{k_1} b_{k_2} k}{k_1 k_2^{2a}} \\ &+ \frac{\tau^{1-2a} U}{(1-2a) L} \sum{}' \frac{b_{k_1} b_{k_2} k}{k_1^{2a} k_2} \log \frac{\tau k_2}{k_1} \\ &- \frac{\tau^{1-2a} U}{(1-2a)^2 L} \sum{}' \frac{b_{k_1} b_{k_2} k}{k_1 k_2^{2a}} + \frac{\tau^{1-2a} U}{(1-2a)^2 L} \sum{}' \frac{b_{k_1} b_{k_2} k}{k_1^{2a} k_2} \\ &+ O(U/L^5). \end{aligned} \tag{8.26}$$

Since $k_2 \leqslant k_1$, k_2 can be replaced by k_m and k_1 by k_M. If $b_{k_1}b_{k_2}$ is left intact and $\sum'$ replaced by $\sum$, the sum doubles to give

$$
\begin{aligned}
2I_{13} = &- \frac{\tau^{1-2a}U}{2(1-2a)} \sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} \\
&+ \frac{\tau^{1-2a}U}{(1-2a)L} \sum \frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m} \log \frac{\tau k_m}{k_M} \\
&- \frac{\tau^{1-2a}U}{(1-2a)^2 L} \sum \frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}} \\
&+ \frac{\tau^{1-2a}U}{(1-2a)^2 L} \sum \frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m} + O\left(\frac{U}{L^5}\right).
\end{aligned}
\tag{8.27}
$$

9. The Evaluation of I_{23}

From (2.27),

$$I_{23} = -\int_T^{T+U} \psi\bar{\psi} g_2 \bar{g}_3 \bar{\chi}_1 \, dt/\log^2(t/2\pi).$$

Proceeding much as for I_{13} instead of (8.16),

$$I_{23} = I_{23,2} + O(U/L^7),$$

where instead of (8.14)

$$I_{23,2} = -\frac{2\pi}{L^2} \sum \frac{b_{k_1}b_{k_2}}{k_2} k_1^{1-2a} \sum_K j_1^{1-2a} \log j_1 \log jA_2 .$$

In place of (8.17), (8.19), (8.20), and (8.21),

$$
\begin{aligned}
&\sum_K j_1^{1-2a} \log j_1 \log jA_2 \\
&\quad = O(L^2T^{1/2}) + \frac{1}{8}\int_{\theta_1}^{\theta_2} (\cos\theta)^{1-2a}\, d\theta \int_{x_1}^{x_2} x^{1/2-a} \log(x\cos^2\theta)\log(x\sin^2\theta A_2{}^2)\, dx \\
&\quad = F_1{}^1 + O(L^2T^{1/2}) + O\left(\frac{U}{A_1L^7}\right),
\end{aligned}
$$

where

$$F_1{}^1 = \frac{\tau^{1-2a}U}{16\pi A_1^{3/2-a}} \int_{1/A_1}^{A_1/A_2{}^2} \log\frac{T}{2\pi A_1 v} \log\frac{TA_2{}^2 v}{2\pi A_1} \frac{dv}{v^{3/2-a}},$$

and as in (8.23),

$$I_{23,3} = -\frac{2\pi}{L^2} \sum{}' \frac{b_{k_1}b_{k_2}}{k_2} k_1^{1-2a} F_1{}^1. \tag{9.1}$$

As in (8.24),

$$I_{23} = I_{23,3} + O(U/L^5). \tag{9.2}$$

By integration by parts,

$$F_1{}^1 = \frac{\tau^{1-2a}U}{16\pi A_1^{3/2-a}}\left\{-\frac{v^{-1/2+a}}{1/2-a}\log\frac{T}{2\pi A_1 v}\log\frac{TA_2{}^2v}{2\pi A_1}\Big|_{1/A_1}^{A_1/A_2{}^2}\right.$$
$$-\frac{v^{-1/2+a}}{(1/2-a)^2}\log\frac{T}{2\pi A_1 v}\Big|_{1/A_1}^{A_1/A_2{}^2}+\frac{v^{-1/2+a}}{(1/2-a)^2}\log\frac{TA_2{}^2v}{2\pi A_1}\Big|_{1/A_1}^{A_1/A_2{}^2}$$
$$\left.+2\frac{v^{-1/2+a}}{(1/2-a)^3}\Big|_{1/A_1}^{A_1/A_2{}^2}\right\}$$
$$=\frac{\tau^{1-2a}U}{16\pi}\left\{-\frac{2A_2^{1-2a}}{(1/2-a)\,A_1^{2-2a}}L\log\frac{\tau k_2}{k_1}\right.$$
$$+\frac{2}{(1/2-a)\,A_1}L\log\frac{\tau k_2}{k_1}-\frac{2A_2^{1-2a}}{(1/2-a)^2\,A_1^{2-2a}}\log\frac{\tau k_2}{k_1}$$
$$+\frac{1}{(1/2-a)^2\,A_1}L+\frac{A_2^{1-2a}}{(1/2-a)^2\,A_1^{2-2a}}L-\frac{2}{(1/2-a)^2\,A_1}\log\frac{\tau k_2}{k_1}$$
$$\left.+\frac{2A_2^{1-2a}}{(1/2-a)^3\,A_1^{2-2a}}-\frac{2}{(1/2-a)^3\,A_1}\right\}.$$

Used in (9.1) with (9.2), this gives the analogue of (8.26) for I_{23} which with $k_2 = k_m$ and $k_1 = k_M$ gives, as with (8.27),

$$2I_{23} = +\frac{\tau^{1-2a}U}{4L(1/2-a)}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\log\frac{\tau k_m}{k_M}$$
$$-\frac{\tau^{1-2a}U}{4L(1/2-a)}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\log\frac{\tau k_m}{k_M}$$
$$+\frac{\tau^{1-2a}U}{4L^2(1/2-a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\log\frac{\tau k_m}{k_M}$$
$$-\frac{\tau^{1-2a}U}{8L(1/2-a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}-\frac{\tau^{1-2a}U}{8L(1/2-a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}$$
$$+\frac{\tau^{1-2a}U}{4L^2(1/2-a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\log\frac{\tau k_m}{k_M}-\frac{\tau^{1-2a}U}{4L^2(1/2-a)^3}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}$$
$$+\frac{\tau^{1-2a}U}{4L^2(1/2-a)^3}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}+O\left(\frac{U}{L^5}\right).$$

10. Evaluation of the Sum of I's

Combining the asymmetric terms (those involving k_m, k_M) in $I_{11} + I_{22} + I_{33} + 2(I_{12} + I_{13} + I_{23})$ results in zero as the following tabulation shows. In this tabulation, the terms associated with each sum are selected in sequence from the successive I's. There are 24 of these terms in all.

$$\frac{\tau^{1-2a}U}{(1-2a)^3}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\left(\frac{2}{L^2}-\frac{2}{L^2}\right)$$
$$+\frac{\tau^{1-2a}U}{(1-2a)^3}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\left(-\frac{2}{L^2}+\frac{2}{L^2}\right)$$
$$+\frac{\tau^{1-2a}U}{(1-2a)^2}\sum\frac{b_{k_2}b_{k_2}k}{k_M k_m^{2a}}\left(-\frac{1}{2L}+\frac{2}{L}-\frac{1}{L}-\frac{1}{2L}\right)$$
$$+\frac{\tau^{1-2a}U}{(1-2a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\left(-\frac{1}{2L}+\frac{1}{L}-\frac{1}{2L}\right)$$
$$+\frac{\tau^{1-2a}U}{(1-2a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\log\frac{\tau k_m}{k_M}\left(-\frac{1}{L^2}+\frac{1}{L^2}\right)$$
$$+\frac{\tau^{1-2a}U}{(1-2a)^2}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\log\frac{\tau k_m}{k_M}\left(-\frac{1}{L^2}+\frac{1}{L^2}\right)$$
$$+\frac{\tau^{1-2a}U}{(1-2a)}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\left(1-\tfrac{1}{2}-\tfrac{1}{2}\right)$$
$$+\frac{\tau^{1-2a}U}{1-2a}\sum\frac{b_{k_1}b_{k_2}k}{k_M k_m^{2a}}\log\frac{\tau k_m}{k_M}\left(\frac{1}{2L}-\frac{1}{L}+\frac{1}{2L}\right)$$
$$+\frac{\tau^{1-2a}U}{1-2a}\sum\frac{b_{k_1}b_{k_2}k}{k_M^{2a}k_m}\log\frac{\tau k_m}{k_M}\left(-\frac{1}{2L}+\frac{1}{L}-\frac{1}{2L}\right)=0.$$

Let

$$S_0=\sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}},\tag{10.1}$$

$$S_1=\sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}}\log\frac{k_1}{k},\tag{10.2}$$

$$S_2=\sum\frac{b_{k_1}b_{k_2}k^{2a}}{k_1^{2a}k_2^{2a}}\log\frac{k_1}{k}\log\frac{k_2}{k},\tag{10.3}$$

$$K_0=\sum\frac{b_{k_1}b_{k_2}k^{2-2a}}{k_1k_2},\tag{10.4}$$

$$K_1 = \sum \frac{b_{k_1} b_{k_2} k^{2-2a}}{k_1 k_2} \log \frac{k_1}{k}, \tag{10.5}$$

$$K_2 = \sum \frac{b_{k_1} b_{k_2} k^{2-2a}}{k_1 k_2} \log \frac{k_1}{k} \log \frac{k_2}{k}. \tag{10.6}$$

Next, combining the symmetric term in the I's shows by (2.21) that

$$\begin{aligned}
&\frac{1}{U}\int_T^{T+U} |\psi H(a+it)|^2\,dt \\
&= S_0\left(\frac{-1}{1-2a} + C_1 - \frac{2}{L^2(1-2a)^3} + \frac{C_4}{L^2} - \frac{2}{L(1-2a)^2} - \frac{C_9}{L}\right) \\
&+ S_1\left(\frac{2}{L^2(1-2a)^2} + \frac{C_3}{L^2} + \frac{2}{L(1-2a)} - \frac{C_8}{L}\right) \\
&+ S_2\left(\frac{-1}{L^2(1-2a)} + \frac{C_2}{L^2}\right) \\
&+ \frac{\tau^{2-4a}}{L^2} K_0\left(\frac{2}{(1-2a)^3} + C_7\right) \\
&+ \frac{\tau^{2-4a}}{L^2} K_1\left(\frac{2}{(1-2a)^2} + C_6\right) + \frac{\tau^{2-4a}}{L^2} K_2\left(\frac{1}{1-2a} + C_5\right) \\
&+ O(1/L^5).
\end{aligned} \tag{10.7}$$

11. Evaluation of S_0

By Lemma 3.8,

$$k^{2a} = \sum_{j|k} j^{2a} F(j, 2a),$$

where

$$F(j, w) = \prod_{p|j}\left(1 - \frac{1}{p^w}\right). \tag{11.1}$$

Thus,

$$\begin{aligned}
S_0 &= \sum \frac{b_{k_1} b_{k_2}}{k_1^{2a} k_2^{2a}} \sum_{j|(k_1,k_2)} j^{2a} F(j, 2a) \\
&= \sum j^{2a} F(j, 2a) \left(\sum_{j|k_1} \frac{b_{k_1}}{k_1^{2a}}\right)^2,
\end{aligned} \tag{11.2}$$

where $j \leqslant y$ and $k_1 \leqslant y$. Let

$$S_{01} = \sum_{j|k_1} \frac{b_{k_1}}{k_1^{2a}} = \sum_{j|k_1} \frac{\mu(k_1) \log y/k_1}{k_1^{1/2+a} \log y}.$$

If $k_1 = nj$, then $n \leqslant y/j$ and

$$S_{01} = \frac{\mu(j)}{j^{1/2+a}} \frac{1}{\log y} S_{02},$$

where

$$S_{02} = \sum{}' \frac{\mu(n) \log y/(nj)}{n^{1/2+a}} \tag{11.3}$$

and $\sum'$ is for $n \leqslant y/j$ and $(n, j) = 1$. Clearly,

$$S_0 = \sum \frac{\mu^2(j) F(j, 2a)}{j \log^2 y} (S_{02})^2. \tag{11.4}$$

Let $x = y/j$. Since $x < T$, $x^{1/2-a} = O(1)$. Using the series for $1/(\zeta F)$,

$$S_{02} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{w-1/2-a}\, dw}{(w - \frac{1}{2} - a)^2 \zeta(w) F(j, w)}. \tag{11.5}$$

The path of integration is now deformed so that

$$S_{02} = Q_0 + Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \tag{11.6}$$

where Q_0 is the residue at $w = \frac{1}{2} + a$; Q_1 is the integral on $w = 1 + iv$, $-\infty < v \leqslant -L^{10}$; Q_2 is the integral on $v = L^{10}$, $1 - b \leqslant u \leqslant 1$; Q_3 is the integral on $u = 1 - b$, $-L^{10} < v < L^{10}$; Q_4 is symmetric to Q_2 and Q_5 to Q_1.

If

$$b = \frac{1}{M \log L},$$

where M is a sufficiently large constant, then [1, p. 53]

$$\left| \frac{1}{\zeta(w)} \right| \leqslant M_1 \log L, \qquad w \in Q_2, Q_3, Q_4$$

for some constant M_1 and

$$\frac{1}{\zeta(1 + iv)} = O(\log |v|), \qquad w \in Q_1, Q_5.$$

Let

$$F_1(j, w) = \prod_{p|j} \left(1 + \frac{1}{p^w}\right).$$

Then recalling (2.1),

$$Q_1, Q_5 = O\left(\frac{1}{(F(j,1))}\right)\int_{L^{10}}^{\infty} \frac{\log v}{v^2}\,dv$$
$$= O\left(\frac{\log L}{L^{10}F(j,1)}\right) = O(\log L F_1(j,1)\,L^{-10}).$$

Similarly,

$$Q_2, Q_4 = O\left(\frac{\log L}{L^{20}F(j,1-b)}\right) = O(\log L F_1(j,1-b)\,L^{-20}),$$
$$Q_3 = O(x^{-b}F_1(j,1-b)\log L)\int_{-L^{10}}^{L^{10}} \frac{dv}{v^2 + (b+\frac{1}{2}-a)^2}$$
$$= O(j^b F_1(j,1-b)\log^2 L/y^b).$$

If $Q_6 = Q_1 + Q_2 + Q_4 + Q_5$, then

$$Q_6 = O(\log L F_1(j,1-b)\,L^{-10}) \tag{11.7}$$

and

$$S_{02} = Q_0 + Q_3 + Q_6. \tag{11.8}$$

To compute Q_0, it is convenient to introduce

$$Z(w) = \frac{1}{(w-1)\,\zeta(w)},$$

where $Z(1) = 1$ and $Z(w)$ is analytic for $|w-1|$ small so that $Z(\frac{1}{2}+a) = 1 + O(\frac{1}{2}-a)$. Clearly,

$$Q_0 = \frac{d}{dw}\left(x^{w-1/2-a}(w-1)\,Z(w)/F(j,w)\right)\Big|_{w=1/2+a}$$
$$= \frac{1}{F(j,\frac{1}{2}+a)}\Big[1 - (\tfrac{1}{2}-a)\log x + O(\tfrac{1}{2}-a)$$
$$+ O((\tfrac{1}{2}-a)^2\log x) + O(\tfrac{1}{2}-a)\sum_{p\mid j}\frac{\log p}{p^{1/2+a}-1}\Big],$$

where use was made of (11.1) for F'/F. Using Lemma 3.9 and $\frac{1}{2} - a = O(1/L)$,

$$Q_0 = [1 - (\tfrac{1}{2}-a)\log(y/j)]/F(j,\tfrac{1}{2}+a) + \frac{1}{L}\,O(F_1(j,\tfrac{1}{2}+a)\log L).$$

This, with (11.8) and (11.7), gives

$$S_{02} = Q_{00} + Q_3 + Q_7 , \tag{11.9}$$

where

$$Q_{00} = [1 - (\tfrac{1}{2} - a) \log(y/j)]/F(j, \tfrac{1}{2} + a)$$

and

$$Q_7 = O(\log L F_1(j, 1 - b)/L).$$

Since $Q_{00} = O(F_1(j, \tfrac{1}{2} + a) = O(F_1(j, 1 - b)$,

$$S_{02}^2 = Q_{00}^2 + O(F_1(j, 1 - b))(Q_3 + Q_7) + O(Q_3^2 + Q_7^2).$$

Therefore, by (11.4),

$$S_0 \log^2 y = P_0 - (1 - 2a) P_1 + (\tfrac{1}{2} - a)^2 P_2 + P_3 + P_4 + P_5 + P_6 , \tag{11.10}$$

where

$$P_m = \sum \frac{\mu^2(j) F(j, 2a)}{jF^2(j, \frac{1}{2} + a)} \log^m \frac{y}{j} , \qquad m = 0, 1, 2, \tag{11.11}$$

$$P_3 = O\left(\frac{\log^2 L}{y^b}\right) \sum \mu^2(j) F_1^2(j, 1 - b) j^{b-1},$$

$$P_4 = O\left(\frac{\log L}{L}\right) \sum \mu^2(j) F_1^2(j, 1 - b)/j,$$

$$P_5 = O\left(\frac{\log^4 L}{y^{2b}}\right) \sum \mu^2(j) F_1^2(j, 1 - b) j^{2b-1},$$

$$P_6 = O\left(\frac{\log^2 L}{L^2}\right) \sum \mu^2(j) F_1^2(j, 1 - b)/j.$$

Let

$$Y(a) = \prod \left(1 + \frac{2p^{1/2-a} - p^{1-2a} - 1/p^{2a}}{p(p + 1)(1 - 1/p^{1/2+a})^2}\right).$$

This is an analytic function of a for $| a - \tfrac{1}{2} | < 1/4$. Therefore, since

$$Y(\tfrac{1}{2}) = \prod \left(1 + \frac{1}{p^2 - 1}\right) = \prod \left(\frac{p^2}{p^2 - 1}\right),$$

$$Y(a) = \prod \left(\frac{p^2}{p^2 - 1}\right) + O((\tfrac{1}{2} - a)). \tag{11.12}$$

By Lemmas 3.11–3.13,

$$P_m = Y(a) \prod (1 - 1/p^2)(\log y)^{m+1}/(m+1) + O(\log^m y), \qquad m = 0, 1, 2,$$

and so by (11.12),

$$P_m = (\log y)^{m+1}/(m+1) + O(\log^m y), \qquad m = 0, 1, 2. \tag{11.13}$$

Let $d_3(n)$ be the coefficient of n^{-s} in $\zeta^3(s)$. Since

$$\left(1 + \frac{1}{p^{1-b}}\right)^2 \leqslant (1 + 3/p^{1-b}),$$

$$\begin{aligned} \sum \mu^2(j) F_1^2(j, 1-b) j^{b-1} &\leqslant \sum \mu^2(j) j^{b-1} \sum_{n|j} d_3(n)/n^{1-b} \\ &\leqslant \sum \frac{d_3(n)}{n^{1-2b}} \sum_{n|j} j^{b-1} \leqslant \sum \frac{d_3(n)}{n^{2-2b}} \sum_{r \leqslant y/n} r^{b-1} \\ &\leqslant O\left(\frac{y^b}{b}\right) \sum \frac{d_3(n)}{n^{2-b}} = O\left(\frac{y^b}{b}\right). \end{aligned}$$

Therefore,

$$P_3 = O(\log^3 L). \tag{11.14}$$

Similarly,

$$P_5 = O(\log^5 L),$$

$$P_4 = O(\log L),$$

$$P_6 = O(\log^2 L/L).$$

Hence, by (11.10),

$$S_0 = \frac{1}{\log y} - \frac{1-2a}{2} + \frac{1}{3}\left(\tfrac{1}{2} - a\right)^2 \log y + O\left(\frac{\log^5 L}{L^2}\right). \tag{11.15}$$

12. Evaluation of S_1

By (10.2),

$$S_1 = \sum \frac{b_{k_1} b_{k_2} k^{2a}}{k_1^{2a} k_2^{2a}} \log \frac{k_1}{k}. \tag{12.1}$$

To use Lemma 3.8, it is necessary to evaluate

$$\begin{aligned} & j^{2a} \sum_{m|j} \frac{\mu(m)}{m^{2a}} \log \frac{k_1 m}{j} \\ &\quad = j^{2a} \log \frac{k_1}{j} F(j, 2a) + j^{2a} \sum_{m|j} \frac{\mu(m)}{m^{2a}} \sum_{p|m} \log p \end{aligned} \tag{12.2}$$

for square-free j. Clearly,

$$\sum_{m|j} \frac{\mu(m)}{m^{2a}} \sum_{p|m} \log p = \sum_{m|j} \frac{\mu(m)}{m^{2a}} \sum_{pr=m} \log p$$
$$= \sum_{pr|j} \frac{\mu(rp)}{(rp)^{2a}} \log p$$
$$= -\sum_{p|j} \frac{\log p}{p^{2a}} \sum_{r|j/p} \frac{\mu(r)}{r^{2a}} = -F(j, 2a) \sum_{p|j} \frac{\log p}{p^{2a}-1}.$$

This and (12.2) in (12.1) yields

$$\log^2 y S_1 = \sum \frac{\mu(k_1)\,\mu(k_2)}{k_1^{1/2+a} k_2^{1/2+a}} \log \frac{y}{k_1} \log \frac{y}{k_2} \sum_{j|(k_1,k_2)} j^{2a} \log \frac{k_1}{j} F(j, 2a) + S_1{}^1, \quad (12.3)$$

where

$$S_1{}^1 = -\sum \frac{\mu(k_1)\,\mu(k_2)}{k_1^{1/2+a} k_2^{1/2+a}} \log \frac{y}{k_1} \log \frac{y}{k_2} \sum_{j|(k_1,k_2)} j^{2a} F(j, 2a) \sum_{p|j} \frac{\log p}{p^{2a}-1}$$
$$= -\sum j^{2a} F(j, 2a) \sum_{p|j} \frac{\log p}{p^{2a}-1} \Big(\sum_{j|k_1} \frac{\mu(k_1)}{k_1^{1/2+a}} \log \frac{y}{k_1} \Big)^2$$
$$= -\sum \frac{\mu^2(j) F(j, 2a)}{j} \sum_{p|j} \frac{\log p}{p^{2a}-1} (S_{02})^2,$$

where S_{02} is as in (11.3). Here the evaluation of S_{02}^2 as above (11.10) can be more crude than before. It suffices to note that

$$Q_{00} = O(F_1(j, 1/2 + a)).$$

By Lemma 3.9,

$$\sum_{p|j} \frac{\log p}{p^{2a}-1} = O(\log L).$$

All this leads much as in the computation of P_3 in S_0 to

$$S_1{}^1 = O(L \log L). \quad (12.4)$$

Inverting the order of summation in (12.3),

$$\log^2 y S_1 = \sum \frac{\mu^2(j) F(j, 2a)}{j} S_{02} S_{12} + O(L \log L), \quad (12.5)$$

where, with $\sum'$ as for S_{02} and $\log n = \log y/j - \log y/jn$,

$$S_{12} = \sum' \frac{\mu(n)}{n^{1/2+a}} \log n \log \frac{y}{nj}$$
$$= S_{02} \log y/j - S_{13}\,, \tag{12.6}$$

where

$$S_{13} = \sum' \frac{\mu(n)}{n^{1/2+a}} \log^2 \frac{y}{nj}$$
$$= \frac{2}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{w-1/2-a}\, dw}{(w-1/2-a)^3\, \zeta(w) F(j, w)}\,,$$

with $x = y/j$ as before. This is evaluated very much like S_{02} by residue-theory and leads in the same way as (11.8) to

$$S_{13} = Q_{10} + Q_3 \log L + Q_6\,, \tag{12.7}$$

where Q_{10} is due to the residue at $w = 1/2 + a$ and is

$$Q_{10} = \frac{d^2}{dw^2} (x^{w-1/2-a}(w-1)\, Z(w)/F(j, w)) \Big|_{w=1/2+a}$$
$$= \frac{1}{F(j, 1/2+a)} \Big\{(2 \log x - (1/2-a) \log^2 x)\, Z(1/2+a)$$
$$+ 2(1-(1/2-a)\log x) \Big(Z'(1/2+a) - Z(1/2+a) \sum_{p|j} \frac{\log p}{p^{1/2+a}-1}\Big)$$
$$- (1/2-a) \Big(Z''(1/2+a) - 2Z'(1/2+a) \sum_{p|j} \frac{p^{1/2+a} \log p}{p^{1/2+a}-1}\Big)$$
$$- (1/2-a)\, Z(1/2+a) \Big(\sum_{p|j} \frac{\log p}{p^{1/2+a}-1}\Big)^2$$
$$- (1/2-a)\, Z(1/2+a) \sum_{p|j} \frac{p^{1/2+a} \log^2 p}{(p^{1/2+a}-1)^2}\Big\}.$$

Using Lemmas 3.9 and 3.10,

$$Q_{10} = \frac{1}{F(j, 1/2+a)} \Big(2 \log \frac{y}{j} - (1/2-a) \log^2 \frac{y}{j}\Big) + O(F_1(j, 1/2+a) \log L).$$

Hence, using this and (12.7) in (12.6) along with (11.9),

$$S_{12} = - \frac{1}{F(j, 1/2+a)} \log \frac{y}{j} + Q_3 L + Q_7 L. \tag{12.8}$$

Thus

$$S_{02}S_{12} = -Q_{00}Q_{11} + O(F_1(j, 1-b)L(Q_3 + Q_7)) + (Q_3^2 + Q_7^2)\,O(L),$$

where Q_{00} is as below (11.9) and

$$Q_{11} = -\frac{1}{F(j, 1/2 + a)} \log \frac{y}{j}.$$

In (12.5) proceeding much as with the later terms of (11.10), this leads to

$$\log^2 y S_1 = \sum \frac{\mu^2(j) F(j, 2a)}{F^2(j, 1/2 + a) j} \left(-\log \frac{y}{j} + (1/2 - a) \log^2 \frac{y}{j}\right) + O(L(\log L)^5).$$

Using the evaluations of P_1 and P_2 as in (11.11) and (11.13), this gives

$$S_1 = -1/2 + (1/2 - a)(\log y)/3 + O((\log^5 L)/L). \qquad (12.9)$$

13. Evaluation of S_2

By (10.3),

$$S_2 = \sum \frac{b_{k_1} b_{k_2} k^{2a}}{k_1^{2a} k_2^{2a}} \log \frac{k_1}{k} \log \frac{k_2}{k}.$$

To make use of Lemma 3.8, it is necessary to calculate

$$\begin{aligned} j^{2a} \sum_{m|j} & \frac{\mu(m)}{m^{2a}} \log \frac{k_1 m}{j} \log \frac{k_2 m}{j} \\ &= j^{2a} F(j, 2a) \log \frac{k_1}{j} \log \frac{k_2}{j} \\ &\quad + j^{2a} \left(\log \frac{k_1}{j} + \log \frac{k_2}{j}\right) \sum_{m|j} \frac{\mu(m)}{m^{2a}} \log m \\ &\quad + j^{2a} \sum_{m|j} \frac{\mu(m)}{m^{2a}} \log^2 m \\ &= j^{2a} F(j, 2a) \left\{\log \frac{k_1}{j} \log \frac{k_2}{j} - \left(\log \frac{k_1}{j} + \log \frac{k_2}{j}\right) \sum_{p|j} \frac{\log p}{p^{2a} - 1}\right. \\ &\quad \left. + \sum_{pq|j} \frac{\log p \log q}{(p^{2a} - 1)(q^{2a} - 1)} - \sum_{p|j} \frac{\log^2 p}{p^{2a} - 1}\right\}, \end{aligned} \qquad (13.1)$$

where q like p is a prime. Note that

$$\sum_{pq|j} \frac{\log p \log q}{(p^{2a} - 1)(q^{2a} - 1)} = \left(\sum_{p|j} \frac{\log p}{p^{2a} - 1}\right)^2 - \sum_{p|j} \frac{\log^2 p}{(p^{2a} - 1)^2}.$$

As was the case with S_1, all but the first term in (13.1) make minor contributions. Thus

$$S_2 \log^2 y = \sum \frac{\mu(k_1)\,\mu(k_2)}{k_1^{1/2+a} k_2^{1/2+a}} \log \frac{y}{k_1} \log \frac{y}{k_2} \sum_{j|(k_1,k_2)} j^{2a} F(j, 2a) \log \frac{k_1}{j} \log \frac{k_2}{j}$$
$$+ O(L^2 \log^2 L)$$
$$= \sum \frac{\mu^2(j)\,F(j, 2a)}{j} S_{12}^2 + O(L^2 \log^2 L), \qquad (13.2)$$

where S_{12} is defined in (12.6). But S_{12} is given by (12.8), and much as in the computation of S_0, this leads in (13.2) to

$$S_2 = (\log y)/3 + O(\log^5 L) \qquad (13.2)$$

14. The Evaluation of the K_j

The evaluation of K_0, K_1, and K_2 is very similar to S_0, S_1, S_2. The only significant difference is that the residues are different. As for $S_0(11.2)$,

$$K_0 = \sum j^{2-2a} F(j, 2-2a) \Big(\sum_{j|k_1} \frac{b_{k_1}}{k_1}\Big)^2,$$

$$\log^2 y K_0 = \sum \frac{\mu^2(j)}{j} F(j, 2-2a)\, K_{02}^2,$$

where, in analogy with S_{02}, (11.3),

$$K_{02} = \sum{}' \frac{\mu(n) \log y/nj}{n^{3/2-a}},$$

and as for (11.5),

$$K_{02} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{w-3/2+a}\, dw}{(w - 3/2 + a)^2\, \zeta(w) F(j, w)}.$$

The residue for K_{02}, in contrast to Q_{00}, has as its significant term

$$[1 + (1/2 - a) \log y/j]/F(j, 3/2 - a). \qquad (14.1)$$

This leads ultimately to

$$K_0 = \frac{1}{\log y} + (1/2 - a) + \tfrac{1}{3}(1/2 - a)^2 \log y + O\left(\frac{\log^5 L}{L^2}\right). \qquad (14.2)$$

For K_1, in place of (12.5),

$$\log^2 y K_1 = \sum \frac{\mu^2(j) F(j, 2-2a)}{j} K_{02} K_{12} + O(L \log L), \qquad (14.3)$$

where K_{02} is as above and

$$K_{12} = \sum{}' \frac{\mu(n)}{n^{3/2-a}} \log n \log \frac{y}{nj}$$
$$= K_{02} \log y/j - K_{13},$$

where

$$K_{13} = \sum{}' \frac{\mu(n)}{n^{3/2-a}} \log^2 \frac{y}{nj}$$
$$= \frac{2}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{w-3/2+a}\, dw}{(w - 3/2 + a)^3 \zeta(w) F(j, w)}.$$

The significant term for K_{13} is

$$[2 \log y/j + (1/2 - a) \log^2 y/j]/F(j, 3/2 - a),$$

and so for K_{12} it is

$$-(\log y/j)/F(j, 3/2 - a). \qquad (14.4)$$

Combining this with (14.1) in (14.3) gives

$$K_1 = -1/2 - (1/2 - a)(\log y)/3 + O(\log^5 L)/L). \qquad (14.5)$$

Finally, for K_2 in place of (13.2),

$$\log^2 y K_2 = \sum \frac{\mu^2(j) F(j, 2-2a)}{j} K_{12}^2 + O(L^2 \log^2 L),$$

and with (14.4), the significant term for K_{12}, this leads to

$$K_2 = (\log y)/3 + O(\log^5 L). \qquad (14.6)$$

15. Proof of the Theorem

Using S_0, S_1, S_2, K_0, K_1, K_2 as just evaluated in (10.7) gives

$$\frac{1}{U}\int_T^{T+U} |\psi H(a+it)|^2\,dt$$
$$= \left[\frac{1}{\log y} - (\tfrac{1}{2}-a) + \tfrac{1}{3}(\tfrac{1}{2}-a)^2 \log y\right]$$
$$\times \left[-\frac{1}{1-2a} - \frac{2}{L(1-2a)^2} - \frac{2}{L^2(1-2a)^3}\right]$$
$$+ [-1/2 + \tfrac{1}{3}(1/2-a)\log y]\left[\frac{2}{L(1-2a)} + \frac{2}{L^2(1-2a)^2}\right] - \frac{\log y}{3L^2(1-2a)}$$
$$+ \left[\frac{1}{\log y} + (1/2-a) + \tfrac{1}{3}(1/2-a)^2 \log y\right]\frac{2\tau^{2-4a}}{L^2(1-2a)^3}$$
$$- [1/2 + \tfrac{1}{3}(1/2-a)\log y]\frac{2\tau^{2-4a}}{L^2(1-2a)^2} + \frac{\log y\,\tau^{2-4a}}{3L^2(1-2a)}$$
$$+ O((\log^5 L)/L).$$

Let $R = (1/2-a)L$. Note that $\log y = L/2 + O(\log L)$. Hence,

$$\frac{1}{U}\int_T^{T+U} |\psi H(a+it)|^2\,dt$$
$$= -\frac{1}{R} + \frac{1}{2} - \frac{R}{12} - \frac{1}{R^2} + \frac{1}{2R} - \frac{1}{12}$$
$$- \frac{1}{2R^3} + \frac{1}{4R^2} - \frac{1}{24R}$$
$$- \frac{1}{2R} + \frac{1}{6} - \frac{1}{4R^2} + \frac{1}{12R} - \frac{1}{12R}$$
$$+ \frac{2e^{2R}}{4R^2}\left(\frac{1}{R} + \frac{1}{2} + \frac{R}{12}\right) - \frac{2e^{2R}}{4R^2}\left(\frac{1}{2} + \frac{R}{6}\right) + \frac{e^{2R}}{12R}$$
$$+ O((\log^5 L)/L) = F(R) + O((\log^5 L)/L),$$

where

$$F(R) = e^{2R}\left(\frac{1}{2R^3} + \frac{1}{24R}\right) - \frac{1}{2R^3} - \frac{1}{R^2} - \frac{25}{24R}$$
$$+ \frac{7}{12} - \frac{R}{12}. \qquad (15.1)$$

Since $F(R)$ is bounded for any given $R \geqslant 0$

$$\int_T^{T+U} |\psi H(a+it)|^2\,dt = O(U)$$

for any finite R and so by (2.20),

$$\int_T^{T+U} |\psi G(a+it)|^2\,dt = UF(R) + O(U\log^5 L/L). \tag{15.2}$$

From (2.5) and (2.6),

$$(1/2 - a)\,2\pi N_G(D) \leqslant \tfrac{1}{2}U\log F(R) + O(U\log^5 L/L). \tag{15.3}$$

If $R = 1.3$, an elementary computation shows that $F(1.3) < 2.3502$, and so for large U, $(1/2 - a)\,2\pi N_G(D) < .4275U$. Since $(1/2 - a)L = 1.3$,

$$2\pi N_G(D) < .3290UL. \tag{15.4}$$

For large t_0, if $1/2 + it_0$ is a zero of $G(s)$, then by (1.7) it is a zero of $\zeta'(s)$ with the same multiplicity, and so by (1.5) it is also a zero of $\zeta(s)$. In D, let N_1 zeros of G be on the left side, $\sigma = 1/2$, and let N_2 be the number of zeros in D with $\sigma > 1/2$. The zeros are counted according to multiplicity.

Indent the rectangle D with small semicircles with centers at the zeros on the left side of D and lying in $\sigma \geqslant 1/2$. Let the number of these zeros, not counting multiplicity, be N_1'. Apply the principle of the argument to the indented D. Let the variation in arg G on the jth interval between the successive semicircles as their radii approach zero be V_j. Recalling that the change in arg G on the right and upper and lower sides of D is dominated by $O(L)$, as shown above (1.12), there results

$$\sum V_j - \pi N_1 = 2\pi N_2 + O(L), \tag{15.5}$$

where $\sum$ is over the intervals separated by the N_1' zeros.

Let W_j be the variation in

$$h(s)[f'(s) + f'(1-s)]\,G(s) \tag{15.6}$$

on the jth interval (in which V_j occurs). W_j is taken for increasing t while V_j is for decreasing t. Recalling (1.2) and (15.5),

$$\sum W_j = \frac{U}{2}L + O\left(\frac{U^2}{T}\right) - (2\pi N_2 + \pi N_1). \tag{15.7}$$

The number of times (1.8) holds on the open jth interval must be at least

$$(W_j/\pi) - 2.$$

Hence, by (15.7), the number of zeros of $\zeta(1/2 + it)$ on the open intervals must be at least

$$\frac{UL}{2\pi} + O\left(\frac{U^2}{T}\right) - (2N_2 + N_1) - 2N_1'$$
$$= \frac{UL}{2\pi} - 2N_G(D) + N_1 - 2N_1' + O\left(\frac{U^2}{T}\right). \qquad (15.8)$$

Now by the remark below (15.4), each zero of $G(s)$ on $\sigma = 1/2$ gives rise to a zero of $\zeta(s)$ with multiplicity one greater. Thus there are $N_1 + N_1'$ such zero of $\zeta(s)$. Adding these to (15.8) gives

$$\frac{UL}{2\pi} - 2N_G(D) + 2N_1 - N_1' + O\left(\frac{U^2}{T}\right)$$

zeros for $\zeta(s)$ on $\sigma = 1/2$. Recall that $N_1 \geqslant N_1'$. By (15.4), the above exceeds

$$.34UL/2\pi$$

and proves the Theorem since

$$N(T + U) - N(T) = \frac{UL}{2\pi} + O\left(\frac{U}{L^{10}}\right).$$

As regards the Corollary to the Theorem, if $\zeta(s)$ has a zero of multiplicity m on $\sigma = 1/2$, then $\zeta'(s)$ has one of multiplicity $m - 1$ and hence so does $G(s)$ by (1.9). The first result of the Corollary now follows from (15.4).

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