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A short proof of Levinson's theorem

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Abstract. We give a short proof of Levinson's result that over 1/3 of the zeros of the Riemann zeta function are on the critical line.

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1. Introduction. In 1974, Levinson [7] proved that 1/3 of the zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line. Apparently his work has a reputation for being difficult, and many textbook authors [4–6,9] present Selberg's method [8] instead (which gives a very small positive percent of zeros). Here we show how innovations in the subject can greatly simplify the proof of Levinson's theorem.

To set some terminology, let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$, and let $N_0(T)$ denote the number of such critical zeros with $\beta = 1/2$. Define κ by $\kappa = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)}$. Levinson's result is that $N_0(T) > \frac{1}{3} N(T)$ for T sufficiently large.

The basic technology to prove that many zeros lie on the critical line is an asymptotic for a mollified second moment of the zeta function (and its derivative). This is well-known, and clear presentations can be found in various sources ([1,7], [9, §10.28].) We briefly summarize the setup. Let $L = \log T$, and suppose $Q(x)$ is a real polynomial satisfying $Q(0) = 1$. Set

$$V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s).$$

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Levinson's original approach naturally had $Q(x) = 1-x$, but Conrey [2] showed how more general choices of Q can be used to improve results. For historical comparison we shall eventually choose $Q(x) = 1 - x$. Let $\sigma_0 = \frac{1}{2} - R/L$ for R a positive real number to be chosen later, $M = T^\theta$ for some $0 < \theta < \frac{1}{2}$, and $P(x) = \sum_j a_j x^j$ be a real polynomial satisfying $P(0) = 0$, $P(1) = 1$. Suppose that ψ is a mollifier of the form

$$\psi(s) = \sum_{h \le M} \frac{\mu(h)}{h^{s+\frac{1}{2}-\sigma_0}} P\left(\frac{\log M/h}{\log M}\right),$$

Again, for historical reasons we eventually take $P(x) = x$. The conclusion is (cf. [9, p. 290])

$$\kappa \ge 1 - \frac{1}{R} \log\left(\frac{1}{T}\int_1^T |V\psi(\sigma_0 + it)|^2 dt\right) + o(1). \tag{1.1}$$

The evaluation of the mollified second moment of zeta appearing in (1.1) is considered to be the difficult part of Levinson's proof (taking up over 30 pages in [7]). Conrey and Ghosh [3] gave a simpler proof. Here we show how to obtain this asymptotic in an easier way.

Theorem 1. *We have*

$$\frac{1}{T}\int_1^T |V\psi(\sigma_0 + it)|^2 dt = c(P, Q, R, \theta) + o(1), \tag{1.2}$$

as $T \to \infty$, where

$$c(P,Q,R,\theta) = 1 + \frac{1}{\theta}\int_0^1\int_0^1 e^{2Rv}\left(\frac{d}{dx}\, e^{R\theta x} P(x+u)Q(v+\theta x)\big|_{x=0}\right)^2 du dv. \tag{1.3}$$

With $P(x) = x$, $Q(x) = 1 - x$, $R = 1.3$, $\theta = .5$, and using any standard computer package,

$$c(P,Q,R,\theta) = 2.35\ldots, \quad \text{and } \kappa \ge 0.34\ldots.$$

2. A smoothing argument. To simplify forthcoming arguments, it is preferable to smooth the integral in (1.2). Suppose that $w(t)$ is a smooth function satisfying the following properties:

$$0 \le w(t) \le 1 \quad \text{for all } t \in \mathbb{R}, \tag{2.1}$$

$$w \text{ has compact support in } [T/4, 2T], \tag{2.2}$$

$$w^{(j)}(t) \ll_j \Delta^{-j}, \text{ for each } j = 0,1,2,\ldots, \quad \text{where } \Delta = T/L. \tag{2.3}$$

Theorem 2. *For any w satisfying (2.1)–(2.3), and $\sigma = 1/2 - R/L$,*

$$\int_{-\infty}^{\infty} w(t)|V\psi(\sigma + it)|^2 dt = c(P,Q,R,\theta)\widehat{w}(0) + O(T/L), \tag{2.4}$$

uniformly for $R \ll 1$, where $c(P,Q,R,\theta)$ is given by (1.3).

We briefly explain how to deduce Theorem 1 from Theorem 2. By choosing w to satisfy (2.1)–(2.3) and in addition to be an upper bound for the characteristic function of the interval $[T/2, T]$, and with support in $[T/2-\Delta, T+\Delta]$, we get

$$\int_{T/2}^{T} |V\psi(\sigma_0+it)|^2 dt \leq c(P,Q,R,\theta)\widehat{w}(0) + O(T/L).$$

Note $\widehat{w}(0) = T/2 + O(T/L)$. We similarly get a lower bound. Summing over dyadic segments gives the full integral.

3. The mean-value results. Rather than working directly with $V(s)$, instead consider the following general integral:

$$I(\alpha,\beta) = \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)|\psi(\sigma_0+it)|^2 dt, \tag{3.1}$$

where $\alpha, \beta \ll L^{-1}$ (with any fixed implied constant). The main result is

Lemma 3. *We have*

$$I(\alpha,\beta) = c(\alpha,\beta)\widehat{w}(0) + O(T/L), \tag{3.2}$$

uniformly for $\alpha, \beta \ll L^{-1}$, *where*

$$c(\alpha,\beta) = 1 + \frac{1}{\theta}\frac{d^2}{dxdy} M^{-\beta x-\alpha y} \int_0^1\int_0^1 T^{-v(\alpha+\beta)} P(x+u)P(y+u)du\Big|_{x=y=0}. \tag{3.3}$$

Proof that Lemma 3 implies Theorem 2. Define I_{smooth} to be the left hand side of (2.4). Then

$$I_{\text{smooth}} = Q\left(-\frac{1}{L}\frac{d}{d\alpha}\right) Q\left(-\frac{1}{L}\frac{d}{d\beta}\right) I(\alpha,\beta)\Big|_{\alpha=\beta=-R/L}. \tag{3.4}$$

We first argue that we can obtain $c(P,Q,R,\theta)$ by applying the above differential operator to $c(\alpha,\beta)$. Since $I(\alpha,\beta)$ and $c(\alpha,\beta)$ are holomorphic with respect to α, β small, the derivatives appearing in (3.4) can be obtained as integrals of radii $\asymp L^{-1}$ around the points $-R/L$, from Cauchy's integral formula. Since the error terms hold uniformly on these contours, the same error terms that hold for $I(\alpha,\beta)$ also hold for I_{smooth}.

Next we check that applying the differential operator to $c(\alpha,\beta)$ does indeed give (1.3). Notice the simple formula

$$Q\left(\frac{-1}{\log T}\frac{d}{d\alpha}\right) X^{-\alpha} = Q\left(\frac{\log X}{\log T}\right) X^{-\alpha}. \tag{3.5}$$

Using (3.5) we have

$$Q\left(-\frac{1}{L}\frac{d}{d\alpha}\right)Q\left(-\frac{1}{L}\frac{d}{d\beta}\right)c(\alpha,\beta) = 1 + \frac{1}{\theta}\frac{d^2}{dxdy}M^{-\beta x-\alpha y}$$
$$\times \int_0^1\int_0^1 T^{-v(\alpha+\beta)}P(x+u)P(y+u)Q(v+x\theta)Q(v+y\theta)dudv\Big|_{x=y=0},$$

which after evaluating at $\alpha=\beta=-R/L$ and simplifying becomes

$$1+\frac{1}{\theta}\frac{d^2}{dxdy}e^{R\theta(x+y)}$$
$$\times\int_0^1\int_0^1 e^{2Rv}P(x+u)P(y+u)Q(v+\theta x)Q(v+\theta y)dudv\Big|_{x=y=0}.$$

This simplifies to give the right hand side of (1.3), as desired. □

4. Two lemmas. A variation on the standard approximate functional equation [5, Theorem 5.3] gives

Lemma 4. *Let $G(s)=e^{s^2}p(s)$ where $p(s)=\frac{(\alpha+\beta)^2-(2s)^2}{(\alpha+\beta)^2}$, and define*

$$V_{\alpha,\beta}(x,t)=\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)x^{-s}ds, \tag{4.1}$$

where

$$g_{\alpha,\beta}(s,t)=\pi^{-s}\frac{\Gamma\left(\frac{\frac12+\alpha+s+it}{2}\right)}{\Gamma\left(\frac{\frac12+\alpha+it}{2}\right)}\frac{\Gamma\left(\frac{\frac12+\beta+s-it}{2}\right)}{\Gamma\left(\frac{\frac12+\beta-it}{2}\right)}. \tag{4.2}$$

Furthermore, set

$$X_{\alpha,\beta,t}=\pi^{\alpha+\beta}\frac{\Gamma\left(\frac{\frac12-\alpha-it}{2}\right)}{\Gamma\left(\frac{\frac12+\alpha+it}{2}\right)}\frac{\Gamma\left(\frac{\frac12-\beta+it}{2}\right)}{\Gamma\left(\frac{\frac12+\beta-it}{2}\right)}.$$

Then if α,β have real part less than $1/2$, and for any $A\geq 0$, we have

$$\zeta\left(\frac12+\alpha+it\right)\zeta\left(\frac12+\beta-it\right)=\sum_{m,n}\frac{1}{m^{\frac12+\alpha}n^{\frac12+\beta}}\left(\frac{m}{n}\right)^{-it}V_{\alpha,\beta}(mn,t)$$
$$+X_{\alpha,\beta,t}\sum_{m,n}\frac{1}{m^{\frac12-\beta}n^{\frac12-\alpha}}\left(\frac{m}{n}\right)^{-it}V_{-\beta,-\alpha}(mn,t)+O_A((1+|t|)^{-A}).$$

Remark. Stirling's approximation gives for t large and s in any fixed vertical strip

$$X_{\alpha,\beta,t}=\left(\frac{t}{2\pi}\right)^{-\alpha-\beta}(1+O(t^{-1})),\quad g_{\alpha,\beta}(s,t)=\left(\frac{t}{2\pi}\right)^{s}(1+O(t^{-1}(1+|s|^2))). \tag{4.3}$$

Furthermore, for any $A \geq 0$ and $j = 0, 1, 2, \ldots$, we have uniformly in x,

$$t^j \frac{\partial^j}{\partial t^j} V_{\alpha,\beta}(x,t) \ll_{A,j} (1 + |t/x|)^{-A}. \tag{4.4}$$

Lemma 5. *Suppose w satisfies (2.1)–(2.3), and that h, k are positive integers with $hk \leq T^{2\theta}$ with $\theta < 1/2$, and $\alpha, \beta \ll L^{-1}$. Then*

$$\int_{-\infty}^{\infty} w(t) \left(\tfrac{h}{k}\right)^{-it} \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)dt$$

$$= \sum_{hm=kn} \frac{1}{m^{\frac{1}{2}+\alpha} n^{\frac{1}{2}+\beta}} \int_{-\infty}^{\infty} V_{\alpha,\beta}(mn,t) w(t) dt$$

$$+ \sum_{hm=kn} \frac{1}{m^{\frac{1}{2}-\beta} n^{\frac{1}{2}-\alpha}} \int_{-\infty}^{\infty} V_{-\beta,-\alpha}(mn,t) X_{\alpha,\beta,t} w(t) dt + O_{A,\theta}(T^{-A}). \tag{4.5}$$

Proof. We apply Lemma 4 to the left hand side. It suffices by symmetry to consider the first part of the approximate functional equation, giving

$$\sum_{m,n} \frac{1}{m^{\frac{1}{2}+\alpha} n^{\frac{1}{2}+\beta}} \int_{-\infty}^{\infty} w(t) \left(\frac{hm}{kn}\right)^{-it} V_{\alpha,\beta}(mn,t) dt.$$

The terms with $hm = kn$ visibly give the first term on the right hand side of (4.5). By combining (2.3) with (4.4), note that we have uniformly in x that

$$\frac{\partial^j}{\partial t^j} w(t) V_{\alpha,\beta}(x,t) \ll_{j,A} (1 + |x/T|)^{-A} \Delta^{-j}.$$

Hence for $hm \neq kn$, we have by repeated integration by parts that

$$\int_{-\infty}^{\infty} w(t) \left(\frac{hm}{kn}\right)^{-it} V_{\alpha,\beta}(mn,t) dt \ll_{j,A} \frac{(1 + \frac{mn}{T})^{-A}}{\Delta^j |\log \frac{hm}{kn}|^j}.$$

Say $hm \geq kn + 1$. Then

$$\left|\log \frac{hm}{kn}\right| \geq \log\left(1 + \frac{1}{kn}\right) \geq \frac{1}{2kn} \geq \frac{1}{2\sqrt{hkmn}}.$$

The same inequality holds in case $kn \geq hm + 1$, by symmetry. The error terms from $hm \neq kn$ are then easily bounded by $O(T^{-A})$ for arbitrarily large A. □

5. Proof of Lemma 3. Inserting the definition of the mollifier ψ, we have

$$I(\alpha,\beta) = \sum_{h,k \leq M} \frac{\mu(h)\mu(k)}{\sqrt{hk}} P\left(\tfrac{\log M/h}{\log M}\right) P\left(\tfrac{\log M/k}{\log M}\right)$$

$$\times \int_{-\infty}^{\infty} w(t) \left(\tfrac{h}{k}\right)^{-it} \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)dt.$$

According to Lemma 5, write $I(\alpha,\beta) = I_1(\alpha,\beta) + I_2(\alpha,\beta) + O(T^{-A})$. Explicitly,

$$I_1(\alpha,\beta) = \sum_{h,k\le M} \frac{\mu(h)\mu(k)}{\sqrt{hk}} P\left(\tfrac{\log M/h}{\log M}\right) P\left(\tfrac{\log M/k}{\log M}\right) \times \sum_{hm=kn} \frac{1}{m^{\frac12+\alpha} n^{\frac12+\beta}} \int_{-\infty}^{\infty} V_{\alpha,\beta}(mn,t) w(t) dt. \tag{5.1}$$

Notice that $I_2(\alpha,\beta)$ is obtained by replacing α with $-\beta$, β with $-\alpha$, and multiplying by $X_{\alpha,\beta,t} = T^{-\alpha-\beta}(1+O(L^{-1}))$. That is, $I(\alpha,\beta) = I_1(\alpha,\beta) + T^{-\alpha-\beta} I_1(-\beta,-\alpha) + O(T/L)$.

Lemma 6. *We have* $I_1(\alpha,\beta) = c_1(\alpha,\beta)\widehat{w}(0) + O(T/L)$, *uniformly on any fixed annuli such that* $\alpha,\beta \asymp L^{-1}$, $|\alpha+\beta| \gg L^{-1}$, *where*

$$c_1(\alpha,\beta) = \frac{1}{(\alpha+\beta)\log M} \frac{d^2}{dxdy} M^{\alpha x+\beta y} \int_0^1 P(x+u)P(y+u)du \Big|_{x=y=0}. \tag{5.2}$$

Remark. Note that $c_1(\alpha,\beta)$ can be alternatively expressed as

$$c_1(\alpha,\beta) = \frac{1}{(\alpha+\beta)\log M} \int_0^1 (P'(u) + \alpha \log M\ P(u))(P'(u) + \beta \log M\ P(u)) du. \tag{5.3}$$

We prove Lemma 6 in Section 6.

Proof that Lemma 6 implies Lemma 3. By adding and subtracting the same thing, we have

$$I(\alpha,\beta) = [I_1(\alpha,\beta) + I_1(-\beta,-\alpha)] + I_1(-\beta,-\alpha)(T^{-\alpha-\beta}-1) + O(T/L).$$

We treat the two terms above differently.

We first compute the term in brackets using (5.3), getting

$$c_1(\alpha,\beta) + c_1(-\beta,-\alpha) = \int_0^1 2P'(u)P(u)du = 1.$$

As for the second term, using (5.2) we see that $(T^{-\alpha-\beta}-1)c_1(-\beta,-\alpha)$ equals

$$\frac{1-T^{-\alpha-\beta}}{(\alpha+\beta)\log M} \frac{d^2}{dxdy} M^{-\beta x-\alpha y} \int_0^1 P(x+u)P(y+u)du \Big|_{x=y=0}.$$

Note that

$$\frac{1-T^{-\alpha-\beta}}{(\alpha+\beta)\log M} = \frac{1}{\theta} \int_0^1 T^{-v(\alpha+\beta)} dv.$$

Gathering the formulas gives (3.3) although with the additional restriction that $|\alpha+\beta| \gg L^{-1}$. However, the holomorphy of $I(\alpha,\beta)$ and $c(\alpha,\beta)$ with $\alpha,\beta \ll L^{-1}$ implies that the error term is also holomorphic in this region.

The maximum modulus principle extends the error term to this enlarged domain. □

6. Proof of Lemma 6. A Mellin formula gives for $1 \le h \le M$ and $i = 1, 2, \ldots$

$$\left(\frac{\log M/h}{\log M}\right)^i = \frac{i!}{(\log M)^i}\frac{1}{2\pi i}\int_{(1)}\left(\frac{M}{h}\right)^v\frac{dv}{v^{i+1}}. \tag{6.1}$$

Using (6.1) and (4.1) in (5.1), we have

$$I_1(\alpha,\beta) = \int_{-\infty}^{\infty} w(t)\sum_{i,j}\frac{a_ia_j i!j!}{(\log M)^{i+j}}\sum_{hm=kn}\frac{\mu(h)\mu(k)}{h^{\frac12}k^{\frac12}m^{\frac12+\alpha}n^{\frac12+\beta}}$$
$$\times\left(\frac{1}{2\pi i}\right)^3\int_{(1)}\int_{(1)}\int_{(1)}\frac{M^{u+v}}{h^vk^u}\frac{g_{\alpha,\beta}(s,t)}{(mn)^s}\frac{G(s)}{s}ds\frac{du\,dv}{u^{i+1}v^{j+1}}.$$

We compute the sum over h, k, m, n as follows

$$\sum_{hm=kn}\frac{\mu(h)\mu(k)}{h^{\frac12+v}k^{\frac12+u}m^{\frac12+\alpha+s}n^{\frac12+\beta+s}}$$
$$= \frac{\zeta(1+u+v)\zeta(1+\alpha+\beta+2s)}{\zeta(1+\alpha+u+s)\zeta(1+\beta+v+s)}A_{\alpha,\beta}(u,v,s), \tag{6.2}$$

where the arithmetical factor $A_{\alpha,\beta}(u,v,s)$ is given by an absolutely convergent Euler product in some product of half planes containing the origin. Next we move the contours to $\mathrm{Re}(u) = \mathrm{Re}(v) = \delta$, and then $\mathrm{Re}(s) = -\delta + \varepsilon$ (for $\delta > 0$ sufficiently small so that the arithmetical factor is absolutely convergent), crossing a pole at $s = 0$ only since $G(s)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2s)$. Since $M \le T^\theta$ with $\theta < \frac12$, and $t \ge T/2$, the new contour of integration gives $O(T^{1-\varepsilon})$ for sufficiently small $\varepsilon > 0$, using (4.3). Thus

$$I_1(\alpha,\beta) = \widehat{w}(0)\zeta(1+\alpha+\beta)\sum_{i,j}\frac{a_ia_ji!j!}{(\log M)^{i+j}}J_{\alpha,\beta}(M) + O(T^{1-\varepsilon}), \tag{6.3}$$

where

$$J_{\alpha,\beta}(M) = \left(\frac{1}{2\pi i}\right)^2\int_{(\varepsilon)}\int_{(\varepsilon)}M^{u+v}\frac{\zeta(1+u+v)A_{\alpha,\beta}(u,v,0)}{\zeta(1+\alpha+u)\zeta(1+\beta+v)}\frac{du\,dv}{u^{i+1}v^{j+1}}.$$

Lemma 7. *We have, uniformly for $\alpha, \beta \ll L^{-1}$,*

$$J_{\alpha,\beta}(M) = \frac{(\log M)^{i+j-1}}{i!j!}\frac{d^2}{dxdy}M^{\alpha x+\beta y}\int_0^1(x+u)^i(y+u)^j\,du\Big|_{x=y=0} + O(L^{i+j-2}). \tag{6.4}$$

Lemma 6 follows directly from Lemma 7 by summing over i and j, and taking a Taylor expansion of $\zeta(1+\alpha+\beta)$.

Proof of Lemma 7. We begin by using the Dirichlet series for $\zeta(1+u+v)$ and reversing the order of summation and integration to get

$$J_{\alpha,\beta}(M) = \sum_{n \leq M} \frac{1}{n} \left(\frac{1}{2\pi i}\right)^2 \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{\left(\frac{M}{n}\right)^{u+v} A_{\alpha,\beta}(u,v,0)}{\zeta(1+\alpha+u)\zeta(1+\beta+v)} \frac{du\, dv}{u^{i+1} v^{j+1}}.$$

Using the standard zero-free region of ζ and upper bound on $1/\zeta$ [see [9], Theorem 3.8 and (3.11.8)], we obtain that $J_{\alpha,\beta}(M)$ equals the residue at $u = v = 0$ plus an error of size

$$\sum_{n \leq M} \frac{1}{n} \left(1 + \log \frac{M}{n}\right)^{-2} \ll 1 \ll L^{i+j-2}.$$

For computing the residue we take contour integrals of radius $\asymp L^{-1}$ and use the Taylor approximation

$$\frac{A_{\alpha,\beta}(u,v,0)}{\zeta(1+\alpha+u)\zeta(1+\beta+v)} = (\alpha+u)(\beta+v)A_{0,0}(0,0,0) + O(L^{-3}).$$

We show in Section 7 below that $A_{0,0}(0,0,0) = 1$, a result we now use freely. Thus $J_{\alpha,\beta}(M)$ equals

$$\sum_{n \leq M} \frac{1}{n} \left(\frac{1}{2\pi i}\right)^2 \left(\oint \left(\frac{M}{n}\right)^u \frac{(\alpha+u)du}{u^{i+1}}\right) \left(\oint \left(\frac{M}{n}\right)^v \frac{(\beta+v)dv}{v^{j+1}}\right) + O(L^{i+j-2}),$$

where the contours are circles of radius 1 around the origin.

We compute these two integrals exactly. Suppose $a > 0$. Then

$$\frac{1}{2\pi i} \oint \frac{a^u(\alpha+u)du}{u^{l+1}} = \frac{d}{dx} e^{\alpha x} \frac{1}{2\pi i} \oint \frac{(ae^x)^u du}{u^{l+1}}\Big|_{x=0} = \frac{1}{l!} \frac{d}{dx} e^{\alpha x} (x + \log a)^l \Big|_{x=0}.$$

Thus $J_{\alpha,\beta}(M)$ equals

$$\frac{1}{i!j!} \frac{d^2}{dx dy} e^{\alpha x + \beta y} \sum_{n \leq M} \frac{1}{n} (x + \log(M/n))^i (y + \log(M/n))^j \Big|_{x=y=0} + O(L^{i+j-2}).$$

Note that

$$\frac{d}{dx} e^{\alpha x} (x + \log(M/n))^i \Big|_{x=0} = \frac{(\log M)^i}{\log M} \frac{d}{dx} M^{\alpha x} \left(x + \frac{\log(M/n)}{\log M}\right)^i \Big|_{x=0},$$

so that by summing over i and j we have

$$J_{\alpha,\beta}(M) = \frac{(\log M)^{i+j-2}}{i!j!} \frac{d^2}{dx dy} \Bigg[M^{\alpha x + \beta y} \times \sum_{n \leq M} \frac{1}{n} \left(x + \frac{\log(M/n)}{\log M}\right)^i \left(y + \frac{\log(M/n)}{\log M}\right)^j \Bigg]_{x=y=0} + O(L^{i+j-2}).$$

By the Euler–Maclaurin formula, we can replace the sum over n by a corresponding integral without introducing a new error term (this requires some thought). That is,

$$J_{\alpha,\beta}(M) = \frac{(\log M)^{i+j-2}}{i!j!} \frac{d^2}{dxdy} \Bigg[M^{\alpha x+\beta y} \times \int_1^M r^{-1} \left(x+\frac{\log(M/r)}{\log M}\right)^i \left(y+\frac{\log(M/r)}{\log M}\right)^j \Bigg]_{x=y=0} + O(L^{i+j-2}).$$

Changing variables $r = M^{1-u}$ and simplifying finishes the proof. □

7. The arithmetical factor. Here we verify that $A_{0,0}(0,0,0) = 1$ as claimed in the proof of Lemma 7. The proof is surprisingly easy. We show that $A_{0,0}(s,s,s) = 1$ for all $\mathrm{Re}(s) > 0$. From (6.2) we have

$$A_{0,0}(s,s,s) = \sum_{hm=kn} \frac{\mu(h)\mu(k)}{(hkmn)^{\frac{1}{2}+s}},$$

noting that the ratios of zeta's on the right hand side of (6.2) cancel. The result now follows instantly from the Möbius formula.

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