NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **MA3201 Rings and Modules**

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Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Other information:

Language: English

Number of pages: 2

Number pages enclosed: 0

Checked by:

Date Signature

- All answers should be justified and properly explained.
- All rings have a multiplicative identity.

Problem 1 Let $\mathbb{F}$ be a field, and let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{F} \right\}$.

a) Show that R is a subring of the ring $M_3(\mathbb{F})$ of 3×3 matrices over $\mathbb{F}$.

Show that $I_1 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b, c \in \mathbb{F} \right\}$ is an ideal of R.

b) Show that I_1 is nilpotent.

Determine whether or not R is a semisimple ring and whether or not R is a left artinian ring.

c) Let $I_2 = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} : b, c, d, e \in \mathbb{F} \right\}$. You may assume that I_2 is an ideal of R.

Determine whether or not R/I_2 is a semisimple ring and whether or not R/I_2 is a left artinian ring.

d) Is I_2 a maximal ideal of R? If not, find the maximal ideals of R containing I_2.

Problem 2 Let R be a ring and M an R-module. Prove that M is cyclic if and only if $M \cong {}_R R/I$ for a left ideal I of R.

Problem 3

a) Find the Smith normal form of the matrix $\begin{pmatrix} 4 & 4 & 4 \\ 2 & 4 & 3 \\ 4 & 4 & 2 \end{pmatrix}$ over $\mathbb{Z}$.

b) Let A be an $n \times n$ matrix over a field $\mathbb{F}$. State without proof how the characteristic polynomial of A and the minimum polynomial[1] of A are related to the invariant factors of $A - xI$ over $\mathbb{F}[x]$.

Let A be a 6×6 matrix over $\mathbb{Q}$ with minimum polynomial $(x^2 - 3x + 2)^2$. Find the possibilities for the invariant factors of $A - xI$ over $\mathbb{Q}[x]$ and compute the rational canonical form of A in one of the cases.

Problem 4 Let R be a ring and let M and N be R-modules.

a) Let $\varphi : M \to N$ be an R-homomorphism. Give the definition of the kernel of φ and show that it is a submodule of M. Show that if φ has an inverse $\varphi^{-1} : N \to M$ then φ^{-1} is an R-homomorphism.

b) Suppose that M and N are simple R-modules. Prove that any R-homomorphism φ from M to N is either zero or an isomorphism.

Let $\mathrm{End}_R(M)$ be the ring of R-homomorphisms from M to M. Prove that $\mathrm{End}_R(M)$ is a division ring.

c) Let n be a positive integer. Show that there is a ring isomorphism

$$\frac{\mathbb{Z}}{n\,\mathbb{Z}} \cong \mathrm{End}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{n\,\mathbb{Z}}\right).$$

Prove that there is exactly one such ring isomorphism.

Prove that if n is not a prime number, then $\mathbb{Z}\,/n\,\mathbb{Z}$ is not a simple $\mathbb{Z}$-module.

[1] The term *minimal polynomial* is used in the textbook for the course: Basic Abstract Algebra, by P. B. Bhattacharya, S. K. Jain and S. R. Nagpaul.