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Faglig kontakt under eksamen:
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EXAM IN RINGS AND MODULES (MA3201)

Thursday, 9th December 2004
Time: 09:00 – 13:00
Grades to be announced: Thursday, 6th January 2005
Permitted aids: None.

Problem 1 Let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{C} \right\}.$$

a) Show that R is a ring under the usual addition and multiplication of matrices.

b) Let

$$I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} \mid b, c, d \in \mathbb{C} \right\}.$$

Show that I is a two-sided ideal in R, and that I is nilpotent.

c) Show that R/I and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic rings. Is R/I a semisimple ring?

d) How can the two-sided ideals in the ring R/I be described in terms of two-sided ideals in R? Find two maximal two-sided ideals in R.

Problem 2

a) Let $\varphi\colon R \to S$ be a homomorphism of rings. Show that any left S-module M becomes a left R-module by defining

$$r \cdot m = \varphi(r)m$$

for all r in R and m in M.

Recall the following: Let F be a field. Suppose A is an algebra over F; that is, there is a map $F \times A \to A$, written $(\alpha, r) \mapsto \alpha \cdot r$, such that A is a vector space over F and

$$\alpha \cdot (rr') = (\alpha \cdot r)r' = r(\alpha \cdot r')$$

for all α in F, and all r and r' in A.

Assume that $0 \neq 1_A$ in A, where 1_A is the identity in A.

b) Show that $\psi \colon F \to A$ given by $\psi(\alpha) = \alpha \cdot 1_A$, is a homomorphism of rings with $\operatorname{Im} \psi \subseteq Z(A)$. Here

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}.$$

Also, show that ψ is injective.

c) Suppose that A is a finite dimensional algebra over F; that is, $\dim_F A$ is finite. Show that A is both left artinian and left noetherian.

Let M be a finitely generated left A-module. Show that M is both an artinian and a noetherian A-module.

Problem 3 Let V be a vector space over a field F with $\dim_F V = n < \infty$. Let $T \colon V \to V$ be a non-zero linear transformation. Then V becomes an $F[x]$-module by letting

$$x^i \cdot v = T^i(v)$$

for all v in V and $i \geq 0$. It is not necessary to prove this.

a) Let $\operatorname{Ann}_{F[x]} V = \{g(x) \in F[x] \mid g(x) \cdot v = 0 \text{ for all } v \in V\}$. Show that $\operatorname{Ann}_{F[x]} V$ is an ideal in $F[x]$.

Let $f(x)$ be the minimal polynomial of T. Show that $\operatorname{Ann}_{F[x]} V = (f(x))$.

b) Suppose that T is a non-zero nilpotent linear transformation; that is, $T^l = 0$ for some positive integer l. Show that the minimal polynomial $f(x)$ of T is equal to x^m for some integer m with $0 < m \leq n$.

c) Suppose also here that T is a non-zero nilpotent linear transformation. What is the smallest possible dimension of the kernel of T? And what is the largest possible dimension of the kernel of T?