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# MA3201 Rings and modules

Monday 13th December 2010
Time: 09:00–13:00
Permitted aids: Simple calculator

**Problem 1** Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 4 & 2 \\ 6 & 1 & -18 & -9 \\ -12 & 0 & 45 & 22 \end{pmatrix}$ in the full matrix ring $M_4(\mathbb{R})$, where $\mathbb{R}$ denotes the real numbers.

**a)** Find the Smith normal form of the matrix

$$A - xI_4 = \begin{pmatrix} 1-x & 0 & 0 & 0 \\ -2 & -x & 4 & 2 \\ 6 & 1 & -18-x & -9 \\ -12 & 0 & 45 & 22-x \end{pmatrix}$$

over $\mathbb{R}[x]$, where $I_4$ denotes identity matrix in $M_4(\mathbb{R})$.

**b)** Compute the rational canonical form of $A$.

**c)** Compute the Jordan canonical form of $A$.

**Problem 2** Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ in the full matrix ring $M_3(\mathbb{Z}_2)$, where $\mathbb{Z}_2$ is the field with two elements.

Define $\psi\colon \mathbb{Z}_2[x] \to M_3(\mathbb{Z}_2)$ by

$$\psi(f(x)) = a_0 I_3 + a_1 A + a_2 A^2 + \cdots + a_m A^m,$$

when $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ in $\mathbb{Z}_2[x]$ and $I_3$ denotes the identity matrix in $M_3(\mathbb{Z}_2)$.

**a)** Show that $\psi$ is a homomorphism of rings.

**b)** Find the kernel $\operatorname{Ker}\psi$ of $\psi$, and show that the image of $\psi$, denoted by $\operatorname{Im}\psi$, is a subring of $M_3(\mathbb{Z}_2)$ and a field with 8 elements.

**c)** Let $F = \operatorname{Im}\psi$. Why is $M_3(\mathbb{Z}_2)$ not an algebra over $F$, when the action of the subring $F$ on $M_3(\mathbb{Z}_2)$ is the natural one? Find a field $k$ such that $M_3(\mathbb{Z}_2)$ is an algebra over $k$, and compute the dimension of $M_3(\mathbb{Z}_2)$ as a vector space over $k$.

**Problem 3** Let $R$ be the subset $\left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ c & 0 & b & 0 \\ 0 & d & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$ of the full $4 \times 4$-matrix ring $M_4(\mathbb{C})$ over the complex numbers $\mathbb{C}$.

**a)** Show that $R$ is a ring with 1.

**b)** Let $I$ be the subset $\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix} \mid c, d \in \mathbb{C} \right\}$ of $R$. Show that $I$ is a two-sided ideal in $R$, and that $R/I \simeq \mathbb{C} \oplus \mathbb{C}$. Is $R$ a semisimple ring?

**c)** (i) Show that $I$ is a nilpotent ideal in $R$.

(ii) Show that any two-sided prime ideal $A$ in $R$ contains any nilpotent two-sided ideal $J$, i.e. $J \subseteq A$ for all nilpotent two-sided ideal $J$ in $R$.

(iii) Find two different maximal two-sided ideals in $R$.

**d)** Let $M$ be a left $R$-module, and let

$$IM = \left\{ \sum_{i=1}^{n} a_i x_i \mid a_i \in I, x_i \in M, \text{ for all } i = 1, 2, \ldots, n,\ n \text{ a positive integer} \right\}.$$

(i) Show that $IM$ is a submodule of $M$.

(ii) Let $I$ be as in (b). The factor $M/IM$ becomes a left module over $R$ via

$$r \cdot (m + IM) = rm + IM,$$

for $r$ in $R$ and $m$ in $M$. Assume that $M/IM$ has $\{\overline{m_1}, \overline{m_2}, \ldots, \overline{m_t}\}$ as generators as a left $R$-module. Let $\{m_i\}_{i=1}^{t}$ be elements in $M$ such that $\overline{m_i} = m_i + IM$ in $M/IM$. Show that $\{m_i\}_{i=1}^{t}$ generates $M$ as left $R$-module.