Problem 1

a) Find the Smith normal form of the matrix $\begin{pmatrix} 2-X & 1 & 2 \\ 0 & 1-X & 2 \\ 1 & 0 & 1-X \end{pmatrix}$ over $\mathbb{Z}_3[X]$.

b) Find the rational canonical form of the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}_3$.

c) Let $M_3(\mathbb{Z}_3)$ be the 3×3 matrix ring over $\mathbb{Z}_3$ and define $\Phi_A : \mathbb{Z}_3[X] \to M_3(\mathbb{Z}_3)$ by letting $\Phi_A(P) = P(A)$ for each polynomial P in $\mathbb{Z}_3[X]$. The Image of Φ_A is then the subring of $M_3(\mathbb{Z}_3)$ generated by the matrix A. Prove that this subring is a field.

Problem 2 Let $\Lambda = \{ \begin{pmatrix} a & b & c \\ b & a & c \\ c & b & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \} \subset M_3(\mathbb{Z}_3)$.

a) Prove that Λ is a commutative subring of $M_3(\mathbb{Z}_3)$, the 3×3 matrix ring over $\mathbb{Z}_3$.

b) Define $\Psi : \Lambda \to \mathbb{Z}_3$ by $\Psi(\begin{pmatrix} a & b & c \\ b & a & c \\ c & b & a \end{pmatrix}) = a + b + c$. Prove that Ψ is a ring homomorphism and find a set of generators for the kernal of Ψ.

c) Is Λ a semisimple ring? You have to give an argument for your answear.

Problem 3 Let R be a ring and A, B and C left R-modules with $A \subseteq B$ and $C \simeq B/A$, (i.e., there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules).

a) Prove that if B is a finitely generated R-module, then C is also finitely generated.

b) Prove that if A and C are both finitely generated, the B is also finitely generated.

c) Prove that B is noetherian if and only if both A and C are noetherian.