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Mid term problems spring 2004

Problem 1

a) In the Euclidean domain $\mathbb{Z}[i]$, prove that $gcd(2+iy, 2-iy) = 1$, when y is an odd integer.

b) Prove that the Diophantine equation

$$y^2 + 4 = z^3$$

has only the integer solutions $y = \pm 11, z = 5$, if y is an odd integer.
(Hint: Factor $y^2 + 4$ in $\mathbb{Z}[i]$.)

Problem 2

a) Prove that if D is a domain that is not a field, then $D[x]$ is not a Euclidean domain.

b) Show that 3 is irreducible, but not prime, in the integral domain $\mathbb{Z}[\sqrt{-5}]$.

Problem 3

a) Find a suitable number $a \in \mathbb{C}$ such that $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(a)$, and find the minimal polynomial of a over $\mathbb{Q}$.

b) Let α be a root of $x^4 - 2 \in \mathbb{Q}[x]$. Write $\dfrac{1}{\alpha^2 - \alpha + 1}$ as $p(\alpha)$, where $p(x) \in \mathbb{Q}[x]$ is a polynomial of degree ≤ 3.

Problem 4

a) Let $x^n - a \in F[x]$ be an irreducible polynomial over F, and let $b \in K$ be a root, where K is an extension field of F. If m is a positive integer such that $m|n$, find the degree of the minimal polynomial of b^m over F.

b) Let D be an integral domain and let F be a subfield of D, such that $[D : F] < \infty$. Prove that D is a field.