Quivers

DEF: A quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph,

$\Gamma_0 = \{\text{vertices}\} = \{1, 2, \dots, n\}$.

$\Gamma_1 = \{\text{arrows}\}$.

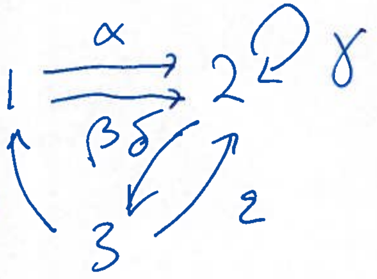
We always assume that Γ_0 and Γ_1 are finite.

Examples

① $\Gamma : 1 \xrightarrow{\alpha} 2$, $\Gamma_0 = \{1,2\}$, $\Gamma_1 = \{\alpha\}$

② $\Gamma : 1 \circlearrowleft \alpha$, $\Gamma_0 = \{1\}$, $\Gamma_1 = \{\alpha\}$

③ Γ :



$\Gamma_0 = \{1,2,3\}$

$\Gamma_1 = \{\alpha, \beta, \gamma, \delta, \varepsilon, \theta\}$.

Have maps: $s, e : \Gamma_1 \longrightarrow \Gamma_0$

$s(\alpha) =$ the vertex where $\alpha \in \Gamma_1$ starts.

$e(\alpha) =$ ——— " ——— ends.

DEF: $\Gamma = (\Gamma_0, \Gamma_1)$ quiver. A path p is either (i) an ordered sequence of arrows $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ where

$$e(\alpha_t) = s(\alpha_{t+1})$$

for $t = 1, 2, \dots, n-1$. (non-trivial path)

$$\cdot \xrightarrow{\alpha_1} \cdot \xrightarrow{\alpha_2} \cdot \xrightarrow{\alpha_3} \cdot \cdots \cdot \xrightarrow{\alpha_n} \cdot$$

or (ii) e_i for each $i \in \Gamma_0$ (trivial path)

$s(p) = s(\alpha_1)$	$s(e_i) = i$
$e(p) = e(\alpha_n)$	$e(e_i) = i$

Examples (1)

$$\begin{tikzcd} \Gamma: 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\gamma"] & 3 \\ & 4 & \end{tikzcd}$$

Paths: (i) $\alpha, \beta, \gamma, \beta\alpha, \gamma\alpha$

(ii) e_1, e_2, e_3, e_4

(2) $\Gamma : 1 \circlearrowleft \alpha$ Paths: (i) $\alpha, \alpha^2 = \alpha\alpha, \alpha^3 = \alpha\alpha\alpha, \dots$

(ii) e_1

Given $\Gamma = (\Gamma_0, \Gamma_1)$ - quiver, k = field

The path algebra $k\Gamma$:

$k\Gamma$ = vector space with all the paths in Γ as a basis.

The elements in $k\Gamma$:

$$a_1 p_1 + a_2 p_2 + \cdots + a_t p_t$$

where $a_i \in k$, p_i paths in Γ.

Example (1)

$$x = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 \alpha + a_6 \beta + a_7 \gamma + a_8 \beta\alpha + a_9 \gamma\alpha$$

$$y = b_1 e_1 + b_2 e_2 + \cdots \quad + b_9 \gamma\alpha$$

$$x + y = (a_1 + b_1) e_1 + (a_2 + b_2) e_2 + \cdots + (a_9 + b_9) \gamma\alpha$$

$$(ap)(bq) \overset{\text{def}}{=} (ab)(p \cdot q) \qquad a, b \in k$$

p, q paths in Γ.

p, q paths in Γ:

(1) p, q ~~both~~ non-trivial

$$p \cdot q \overset{\text{def}}{=} \begin{cases} pq, & \text{if } e(q) = s(p) \\ 0, & \text{otherwise.} \end{cases}$$

(2) p non-trivial, q trivial, $q = e_i$

$$p \cdot q = \begin{cases} p, & \text{if } s(p) = i = e(q) \\ 0, & \text{otherwise} \end{cases}$$

$$q \cdot p = \begin{cases} p, & \text{if } e(p) = i = s(q) \\ 0, & \text{otherwise.} \end{cases}$$

(3) $p = e_i$, $q = e_j$ (both trivial)

$$p \cdot q = \begin{cases} e_i, & \text{if } e(q) = j = i = s(p) \\ 0, & \text{otherwise} \end{cases}$$

This is extended distributively to an operation on $k\Gamma$. (page 50).

Examples (1) $\Gamma: 1 \xrightarrow{\alpha} 2$, k field

Elt's in $k\Gamma$: $a_1 e_1 + a_2 e_2 + a_3 \alpha = y$

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

$$(e_1 + e_2) \cdot y$$

$$(e_1 + e_2)(a_1 e_1 + a_2 e_2 + a_3 \alpha)$$

$$= a_1 e_1^2 + \cancel{a_2 e_1 e_2} + \cancel{a_3 e_1 \alpha} + \cancel{a_1 e_2 e_1} + a_2 e_2^2 + a_3 e_2 \alpha$$

$$= a_1 e_1 + a_2 e_2 + a_3 \alpha = y$$

Similarly: $y(e_1+e_2)=y$

Hence, e_1+e_2 acts like 1 in $k\Gamma$.

Basis for $k\Gamma$: $\{e_1, e_2, \alpha\}$, $\dim_k k\Gamma = 3$

(2) Γ $1 \circlearrowright \alpha$, k field

$k\Gamma$ has basis: $\{e_1, \alpha, \alpha^2, \alpha^3, \dots\}$, i.e. $\dim_k k\Gamma = \infty$

Elt's: $a_0 e_1 + a_1\alpha + a_2\alpha^2 + \cdots + a_t\alpha^t$, $a_i \in k$, $t \geq 0$.

Note: (1) In general, $\{e_i\}_{i\in\Gamma_0}$ are orthogonal idempotents in $k\Gamma$, i.e. $\begin{cases} e_i^2 = e_i \\ e_i e_j = 0 \text{ for } i \neq j \end{cases}$

(2) Suppose $\Gamma_0 = \{1, 2, \dots, n\}$. Then $e_1+e_2+\cdots+e_n$ acts like 1 in $k\Gamma$. Enough to show that $p = (e_1+e_2+\cdots+e_n)p = p(e_1+e_2+\cdots+e_n)$ for any path p. Suppose that $s(p)=i$ and $e(p)=j$. Then

$$(e_1+e_2+\cdots+e_n)p = e_1p + e_2p + \cdots + e_jp + \cdots + e_np$$
$$= e_jp \overset{\text{def}}{=} p$$

$$p(e_1+e_2+\cdots+e_n) = pe_1+pe_2+\cdots+pe_i+\cdots+pe_n$$
$$= pe_i \overset{\text{def}}{=} p$$

$\Longrightarrow e_1+e_2+\cdots+e_n = 1_{k\Gamma}$ = identity in $k\Gamma$.

Can show: $k\Gamma$ is a k-algebra with $e_1+e_2+\cdots+e_n$ as an identity (see page 50 for details

Recall: Λ ring, k field

DEF: Λ is a k-algebra, if Λ is a vector space over k ($k\times\Lambda \longrightarrow \Lambda$, Λ is a module over k, $\alpha\in k$, $\lambda\in\Lambda$, $\alpha\cdot\lambda$) and

$$\alpha(\lambda\cdot\lambda') = (\alpha\cdot\lambda)\lambda' = \lambda(\alpha\cdot\lambda')$$

$\forall \alpha\in k$, $\forall \lambda, \lambda' \in \Lambda$.

Equivalent: Λ is a k-algebra, if $\exists \varphi: k \longrightarrow \Lambda$ a ring homomorphism such that $\operatorname{Im}\varphi \subseteq Z(\Lambda) = \{z\in\Lambda \mid z\lambda = \lambda z, \forall \lambda\in\Lambda\}$.

($\Longleftrightarrow \exists R \subseteq \Lambda$ subring such that $R \simeq k$ with $R \subseteq Z(\Lambda)$)

$\varphi(a) = a\cdot 1_\Lambda$

For $k\Gamma$ the ring hom. $\varphi : k \longrightarrow k\Gamma$ is given by $\varphi(a) = ae_1 + ae_2 + \cdots + ae_n$.

Exercises: (1) $\Gamma : 1 \xrightarrow{\alpha} 2$, k field.

Find a k-algebra isomorphism
$$\psi : k\Gamma \longrightarrow \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

(2) $\Gamma : 1 \circlearrowleft \alpha$, k field

Show that $k\Gamma \simeq k[x]$ as k-algebra's.

DEF: A non-trivial path p in Γ is an oriented cycle if $e(p) = s(p)$.

Example Γ :

Cycles: $\alpha, \alpha^3, \gamma\beta\alpha, \beta\alpha^{10}\gamma, \ldots\ldots$

$\dim_k k\Gamma = \infty$.

Proposition 1 $\Gamma = (\Gamma_0, \Gamma_1)$ quiver, k field.

$\dim_k k\Gamma < \infty \iff \Gamma$ has no oriented cycles.

Proof: Exercise. $\square$

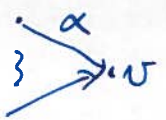
Proposition 2: Assume that $\Gamma = (\Gamma_0, \Gamma_1)$ has no oriented cycles.

$k\Gamma$ is semisimple $\iff \Gamma_1 = \emptyset$.

Proof: Prop 1 $\rightarrow \dim_k k\Gamma < \infty$
$\Rightarrow k\Gamma$ is a left artinian ring.

$k\Gamma$ semisimple $\iff \left\{ \begin{array}{l} \text{no non-zero nilpotent} \\ \text{left ideals in } k\Gamma \end{array} \right.$

$\Rightarrow$: Assume that $\Gamma_1 \neq \emptyset$. Let α_1 be an arrow in Γ. Want to find a vertex where at least one arrow ends and no arrow starts. If $e(\alpha_1)$ is such a vertex, we are done. If not, there is an arrow α_2 starting in $e(\alpha_1)$. If also $e(\alpha_2)$ is not as above, we continue. Since Γ has no oriented cycles and Γ is finite, we must end up in a vertex v, where arrows only end and no arrow starts. Say, $\alpha = \alpha_t$ is an arrow ending in v.



Then consider $k\Gamma\alpha = k\alpha$

since $(a_1\alpha)(a_2\alpha) = (a_1 a_2)\underset{0}{\underset{\shortparallel}{(\alpha\cdot\alpha)}}$

$= 0$

⇒ $(k\Gamma\alpha)^2 = (0)$ and $k\Gamma\alpha \neq (0)$.

⇒ $k\Gamma$ is not semisimple.

<u>⇐</u>: Assume that $\Gamma_1 = \emptyset$. Then

Γ 1 2 $- - - -$ n (n vertices)

<u>Basis for $k\Gamma$:</u> $\{e_1, e_2, \dots, e_n\}$

Elt's in $k\Gamma$: $a_1e_1 + a_2e_2 + \cdots + a_ne_n$

with $a_i \in k$. Have a ring hom.

$$\psi : \underbrace{k\times\cdots\times k}_{n} \longrightarrow k\Gamma$$

given by $\psi(a_1, a_2, \dots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n$

(Check this!). Show that ψ is an isom.

Therefore $k\Gamma$ is semisimple, since $k\Gamma$ is isomorphic to a finite product of full matrix rings over division rings. □

$k\Gamma$ is not always semisimple, but some factor of $k\Gamma$ is.

<u>Proposition 3</u>

$\Gamma = (\Gamma_0, \Gamma_1)$ quiver, k field.

Let $J = \{$all linear comb. of non-trivial paths$\}$

Then J is an ideal in $k\Gamma$ and

$$k\Gamma/J \simeq \underbrace{k\times\cdots\times k}_{|\Gamma_0|}, \text{ - semisimple.} \qquad |\Gamma_0| = n$$

"<u>Proof</u>": Define $\psi : k\Gamma \longrightarrow \overbrace{k\times\cdots\times k} = k^n$

$\psi(a_1e_1 + a_2e_2 + \cdots + a_ne_n + \text{lin. comb. of non-trivial paths})$

$= (a_1, a_2, \dots, a_n)$

<u>Check</u>: (1) ψ is well-defined

(2) ψ homomorphism of rings

(3) $\operatorname{Ker}\psi = J$.

⇒ $k\Gamma/J \simeq \operatorname{Im}\psi = k^n$. □

Modules

Example

$\Gamma : 1 \xrightarrow{\alpha} 2$, k field

What is a module over $k\Gamma$?

Let M be a left $k\Gamma$-module.

Recall: $1_{k\Gamma} = e_1 + e_2$, $e_i e_j = \begin{cases} e_i, & i=j \\ 0, & i \neq j \end{cases}$

Claim: $M = e_1 M \oplus e_2 M$ as a vector space over k.

Proof: $m = 1_{k\Gamma} \cdot m = (e_1 + e_2) \cdot m = e_i m + e_2 m$
$\in e_1 M + e_2 M$

$\Rightarrow M \subseteq e_1 M + e_2 M \subseteq M \Rightarrow M = e_1 M + e_2 M$

Let $m \in e_1 M \cap e_2 M$, i.e. $m = e_1 m' = e_2 m''$

$$e_1 m = e_1(e_1 m') = (e_1 e_1) m' = e_1 m' = m$$

$$\overset{\|}{e_1(e_2 m'')} = \underbrace{(e_1 e_2)}_{\overset{\|}{0}} m'' = 0 \cdot m'' = 0.$$

$\Longrightarrow m = 0$. Hence $e_1 M \cap e_2 M = (0)$.

$\Longrightarrow M = e_1 M \oplus e_2 M$.

Let $m \in M$. Then $e_1 m = e_1(e_1 m + e_2 m)$
$= e_1(e_1 m) + e_1(e_2 m)$
$= e_1^2 m + (e_1 e_2) m$
$= e_1 m$

$e_2 m = e_2(e_1 m + e_2 m) = e_2 m$

$\alpha m = \alpha(e_1 m + e_2 m) = \alpha(e_1 m) + \alpha(e_2 m)$
$= (\alpha e_1) m + \underset{\overset{\|}{0}}{\cancel{(\alpha e_2)}} m$
$= \alpha m = \alpha e_1 m$ (Know: $e_2 \alpha = \alpha$)
$= (e_2 \alpha) \cdot e_1 m = e_2(\alpha e_1 m) \in e_2 M$.

$M \xrightarrow{\alpha \cdot -} M \rightsquigarrow$ linear map $e_1 M \xrightarrow{\alpha \cdot -} e_2 M$

$M \xrightarrow{e_1 \cdot -} M \rightsquigarrow$ linear map, projection $M \longrightarrow e_1 M$

$M \xrightarrow{e_2 \cdot -} M \rightsquigarrow$ projection $M \longrightarrow e_2 M$.

linear map.

$$e_1 M \xrightarrow{\alpha \cdot -} e_2 M$$

A representation of Γ over k.

two vector spaces

Given $V \xrightarrow{f} V'$, V, V' vector spaces over k
f linear map.

How can be construct a left $k\Gamma$-module?

From above: $M = V \oplus V'$ as a vector space

$m = (v, v')$, $e_1 m \overset{\text{def}}{=} (v, 0)$

$e_2 m \overset{\text{def}}{=} (0, v')$

$\alpha = e_2 \alpha e_1$ $\quad \alpha m \overset{\text{def}}{=} (0, f(v))$

<u>Check</u>: M becomes a left $k\Gamma$-module!

<u>DEF</u>: A representation (V, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k is a collection of vector spaces $\{V(i)\}_{i \in \Gamma_0}$ over k and k-linear maps $f_\alpha : V(i) \longrightarrow V(j)$ for each arrow $\alpha : i \to j$ in Γ_1. (We assume that $\dim_k V(i) < \infty$ for all $i \in \Gamma_0$).

<u>Examples</u>

(1) $\Gamma : 1$. A representation of Γ over k is just a vector space over k.

(2) $\Gamma : 1 \xrightarrow{\alpha} 2$

Representation: $V(1) \xrightarrow{f_\alpha} V(2)$

For ex: $k \xrightarrow{1} k$, $k \xrightarrow{0} 0$, $0 \xrightarrow{0} k$

$$k^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix}} k^3$$

(3) Γ:

$$\begin{tikzcd}
 & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\
 & 4 &
\end{tikzcd}$$

Representations:

$$\begin{tikzcd}
 & V(1) \arrow[dl, "f_\alpha"'] \arrow[dr, "f_\beta"] & \\
V(2) \arrow[dr, "f_\gamma"'] & & V(3) \arrow[dl, "f_\delta"] \\
 & V(4) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & k^2 \arrow[dl, "(1\ 0)"'] \arrow[dr, "\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}"] & \\
k \arrow[dr, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"] \\
 & k^2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
 & k &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[dr, "1"'] & & k \arrow[dl, "0"] \\
 & k &
\end{tikzcd}$$

Maps between representations

Example

$\Gamma: 1 \xrightarrow{\alpha} 2$, k field

Let $f: M \to N$ be a homomorphism of left $k\Gamma$-modules. Then

$$f(e_1m) = f((e_1e_1)m) = f(e_1(e_1m))$$
$$= e_1 f(e_1m) \in e_1N$$

$\longrightarrow f|_{e_1M}: e_1M \longrightarrow e_1N$

Similarly, $f|_{e_2M}: e_2M \longrightarrow e_2N$.

Furthermore,

$$\begin{array}{ccc} \alpha\cdot f(e_1m) & = & f(\alpha\cdot(e_1m)) \\ \| & & \| \\ \alpha\cdot f|_{e_1M}(e_1m) & & f|_{e_2M}(\alpha\cdot(e_1m)) \end{array} \qquad \alpha = e_2\alpha$$

Hence

$$\begin{tikzcd} e_1M \arrow[r, "f|_{e_1M}"] \arrow[d, "\alpha\cdot -"'] \arrow[dr, phantom, "\circlearrowright"] & e_1N \arrow[d, "\alpha\cdot -"] \\ e_2M \arrow[r, "f|_{e_2M}"] & e_2N \end{tikzcd}$$

Note: $f \begin{Bmatrix} 1-1 \\ \text{onto} \\ \text{isom.} \end{Bmatrix} \iff f|_{e_iM} \begin{Bmatrix} 1-1 \\ \text{onto} \\ \text{isom.} \end{Bmatrix}$ for all i

DEF: Let (V,f) and (V',f') be two representations of Γ over k. A homomorphism $h:(V,f) \to (V',f')$ is a collection of linear maps

$$h(i): V(i) \longrightarrow V'(i)$$

for all $i \in \Gamma_0$, such that $\forall \alpha: i \to j \in \Gamma_1$ the following diagram commutes

$$\begin{tikzcd} V(i) \arrow[r, "h(i)"] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & V'(i) \arrow[d, "f'_\alpha"] \\ V(j) \arrow[r, "h(j)"] & V'(j) \end{tikzcd}$$

i.e. $f'_\alpha h(i) = h(j) f_\alpha$ for all $\alpha \in \Gamma_1$.

h is a(n) $\begin{Bmatrix} \text{isomorphism} \\ \text{monomorphism} \\ \text{epimorphism} \end{Bmatrix}$ if all $h(i): V(i) \to V'(i$ are $\begin{Bmatrix} \text{isomorphisms} \\ \text{monomorphisms} \\ \text{epimorphisms} \end{Bmatrix}$.

Examples

(1) $\Gamma: 1 \xrightarrow{\alpha} 2$, k field.

(a)
$$\begin{tikzcd}
k \arrow[r, dashed, "a\cdot -"] \arrow[d, "1"'] & k \arrow[d, "0"] \\
k \arrow[r, dashed, "0"] \arrow[d, phantom, "\shortparallel" sloped] & 0 \arrow[d, phantom, "\shortparallel" sloped] \\
(V,f) & (V',f')
\end{tikzcd}$$

$h(1) = a\cdot -$, $a \in k$

$h(2) = 0$

$h = (a\cdot -, 0)$

(b)
$$\begin{tikzcd}
k \arrow[r, dashed, "0"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & 0 \arrow[d, "0"] \\
k \arrow[r, dashed, "0"] \arrow[d, phantom, "\shortparallel" sloped] & k \arrow[d, phantom, "\shortparallel" sloped] \\
(V,f) & (V',f')
\end{tikzcd}$$

No non-zero homomorphisms

(c)
$$\begin{tikzcd}
k^2 \arrow[r, dashed, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0&1\\1&0\end{pmatrix}"'] & k^2 \arrow[d, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"] \\
k^2 \arrow[r, dashed, "\begin{pmatrix}0&1\\1&0\end{pmatrix}"] \arrow[d, phantom, "\shortparallel" sloped] & k^2 \arrow[d, phantom, "\shortparallel" sloped] \\
(V,f) & (V',f')
\end{tikzcd}$$

$h = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$

is an isomorphism.

(2) Γ
$$\begin{tikzcd}
& 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\
& 4 &
\end{tikzcd}$$

$$\begin{tikzcd}
& k \arrow[dl, "1"'] \arrow[dr, "1"] & & & & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \arrow[rrrr, bend right] & & k \arrow[dr, "1"'] & & k \arrow[dl, "0"] \\
& k \arrow[rrrr, bend right, "0"'] \arrow[d, phantom, "\shortparallel" sloped] & & & & k \arrow[d, phantom, "\shortparallel" sloped] & \\
& (V,f) & & & & (V',f') &
\end{tikzcd}$$

$\Longrightarrow (V,f) \not\cong (V',f')$.

Modules $\longleftrightarrow$ representations

$\Gamma = (\Gamma_0, \Gamma_1)$ - quiver, k field.

M left $k\Gamma$-module $\rightsquigarrow$ (V, f) representation of Γ

- $V(i) = e_i M$
- $\alpha : i \to j \in \Gamma_1$

$$f_\alpha : V(i) = e_i M \xrightarrow{\alpha \cdot -} e_j M = V(j)$$

$$e_i m \longmapsto \alpha \cdot e_i m$$

$M = \bigoplus_{i \in \Gamma_0} V(i)$ $\leftsquigarrow$ (V, f) representation of Γ

$$\left.\begin{array}{l} m = (v_1, v_2, \ldots, v_n) \\ e_i m \overset{\text{def}}{=} (0, \ldots, 0, v_i, 0, \ldots, 0) \\ \alpha : i \to j \in \Gamma_1 \quad (\alpha = e_j \alpha e_i) \\ \alpha \cdot m \overset{\text{def}}{=} (0, \ldots, 0, \underset{\uparrow\, j\text{-th coord.}}{f_\alpha(v_i)}, 0, \ldots, 0) \end{array}\right\}$$

Can show: This induces a left $k\Gamma$-module structure on M. (See the ~~book~~ page 57)

Examples (1) $\Gamma : 1 \xrightarrow{\alpha} 2$, k field

$(V, f) : k \xrightarrow{1} k \rightsquigarrow M = k \oplus k = k^2$

$$e_1(a, b) = (a, 0)$$

$$e_2(a, b) = (0, b) \qquad \varphi(a, b) = ae_1 + b\alpha$$

$$\alpha(a, b) = (0, a)$$

$k\Gamma e_1 = k\{e_1, \alpha\}$

Define $\varphi : M \longrightarrow k\Gamma e_1$ by letting $\varphi(1, 0) = e_1$ and $\varphi(0, 1) = \alpha$

Have: $\alpha\varphi(a, b) = \alpha(ae_1 + b\alpha) = a\underset{\alpha}{\underset{\shortparallel}{\alpha e_1}} + b\underset{0}{\underset{\shortparallel}{\alpha^2}}$

$$= a\alpha$$

$$= \varphi(0, a) = \varphi(\alpha(a, b))$$

Similarly, $e_i\varphi(a, b) = \varphi(e_i(a, b))$

$\Rightarrow \varphi$ is a $k\Gamma$-homomorphism.

$$\left.\begin{array}{l} \operatorname{Ker}\varphi = (0) \\ \operatorname{Im}\varphi = k\Gamma e_1 \end{array}\right\} \Rightarrow M \cong k\Gamma e_1 \text{ as a left } k\Gamma\text{-module.}$$

(2) Γ:

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$

, k field

(V,f):

$$\begin{tikzcd} & k \arrow[dl, "f_\alpha = 1"'] \arrow[dr, "1"] & \\ k \arrow[dr, "f_\gamma = \binom{1}{0}"'] & & k \arrow[dl, "\binom{0}{1}"] \\ & k^2 & \end{tikzcd}$$

$$M = V(1) \oplus V(2) \oplus V(3) \oplus V(4) = k \oplus k \oplus k \oplus k^2$$

$$\alpha(v_1, v_2, v_3, v_4) = (0, v_1, 0, 0)$$

$$\gamma(v_1, v_2, v_3, v_4) = (0, 0, 0, (v_2, 0))$$

$$\gamma\alpha(v_1, v_2, v_3, v_4) = (0, 0, 0, (v_1, 0))$$

Exercise: Show that $M \simeq k\Gamma e_1$ as a left $k\Gamma$-module.

Special representations

- Zero representation: $V(i) = (0)$, $\forall i \in \Gamma_0$, $f_\alpha = 0$, $\forall \alpha \in \Gamma_1$
- For each $i \in \Gamma_0$, we have a rep. T_i given by
$$T_i(j) = \begin{cases} k, & \text{if } j = i \\ (0), & \text{otherwise.} \end{cases}$$
and $f_\alpha = 0$ for all $\alpha \in \Gamma_1$.

T_i corresponds to a left $k\Gamma$-module S_i: $S_i \simeq k$ as a vector space.

$$e_j v = \begin{cases} v, & j = i \\ 0, & \text{otherwise} \end{cases} \quad \Bigg| \quad \alpha \cdot v = 0, \ \forall \alpha \in \Gamma_1$$

Recall: Λ k-algebra, k field.

M left Λ-module $\Longrightarrow$ M k-vector space

$\cup$| submodule $\Longrightarrow$ $N \subseteq M$ subspace

N

Note: $\dim_k S_i = 1 \Rightarrow S_i$ is a simple $k\Gamma$-module.

DEF: Λ ring, $(0) \neq M$ left Λ-module. M is indecomposable if $M \simeq M_1 \oplus M_2$ implies that $M_1 = (0)$ or $M_2 = (0)$.

DEF: Let $V = (V, f)$ and $V' = (V', f')$ be two representations of a quiver Γ. Define $W = (W, h) = V \oplus V'$ by

$$W(i) = V(i) \oplus V'(i)$$

and

$$h_\alpha = f_\alpha \oplus f'_\alpha : W(i) = V(i) \oplus V'(i) \longrightarrow V(j) \oplus V'(j) = W(j)$$

for all $\alpha : i \longrightarrow j \in \Gamma_1$.

DEF: $(0) \neq V = (V, f)$ is an indecomposable representation if $V = V_1 \oplus V_2$ implies that $V_1 = (0)$ or $V_2 = (0)$

Example $\Gamma : 1 \xrightarrow{\alpha} 2$, k field

- $k^2 \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} k^2 \cong (k \xrightarrow{1} k) \oplus (k \xrightarrow{1} k)$
- $k \xrightarrow{1} k$ indecomposable? Others?

$\Gamma = (\Gamma_0, \Gamma_1)$ quiver, k field

M, N $k\Gamma$-modules, $N \subseteq M$ submodule

$\Rightarrow$ $e_i N \subseteq e_i M$ subspace

Given $\alpha : i \longrightarrow j \in \Gamma_1$,

$$\begin{tikzcd}
e_iN \arrow[r, hook] \arrow[d, "\alpha\cdot-"'] \arrow[dr, phantom, "\circlearrowright"] & e_iM \arrow[d, "\alpha\cdot-"] \\
e_jN \arrow[r, hook] & e_jM
\end{tikzcd}$$

DEF: $(V, f) \subseteq (V', f')$ is a subrepresentation if

(i) $V(i) \subseteq V'(i)$ subspace $\forall i \in \Gamma_0$,

(ii)
$$\begin{tikzcd}
V(i) \arrow[r, hook] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & V'(i) \arrow[d, "f'_\alpha"] \\
V(j) \arrow[r, hook] & V'(j)
\end{tikzcd}$$
for $\alpha : i \longrightarrow j \in \Gamma_1$ i.e. $f_\alpha = f'_\alpha|_{V(i)}$.

Factor of two representation, $W = V'/V$

- $W(i) = V'(i)/V(i)$

$$\begin{tikzcd}
 & v' \arrow[r, mapsto] \arrow[ddd, bend left, mapsto] & v' + V(i) \\
V(i) \arrow[r, hook] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & V'(i) \arrow[r] \arrow[d, "f'_\alpha"] & V'(i)/V(i) \arrow[d, dashed, "f''_\alpha"] \\
V(j) \arrow[r, hook] & V'(j) \arrow[r] & V'(j)/V(j) \\
 & f'_\alpha(v') \arrow[r, mapsto] & f'_\alpha(v') + V(j)
\end{tikzcd}$$

$$f''_\alpha(v' + V(i)) = f'_\alpha(v') + V(j)$$

Check: f''_α is well-defined.

(W, f'') is a representation of Γ over k.

$$\begin{tikzcd} (V,f) \arrow[r, hook] \arrow[d, squiggly] & (V',f') \arrow[d, squiggly] \\ M_V \arrow[r, hook] & M_{V'} \end{tikzcd} \qquad \begin{tikzcd} W = V'/V \arrow[d, squiggly] \\ M_W \simeq M_{V'}/M_V \end{tikzcd}$$

DEF: Λ finite dimensional k-algebra, k field. Then Λ is of finite representation type if there is only a finite number of non-isomorphic indecomposable finitely generated left Λ-modules.

Examples

(1) $\Delta = k$. The only indecomposable Λ-module is k

(2) $\Gamma : 1 \xrightarrow{\alpha} 2$, k field

The indecomposable left Γ-modules

⟺ The indecomposable representations of Γ over k.

(V,f)

Let $V_1 \xrightarrow{f} V_2$ is an indec. rep. of Γ over k.

Know: $h_1 : V_1 \xrightarrow{\sim} \operatorname{Im} f \oplus \operatorname{Ker} f$ ⑬

In particular $V \underset{f'}{\overset{f}{\rightleftarrows}} \operatorname{Im} f$

such that $ff' = 1_{\operatorname{Im} f}$

$$h_1 : V_1 \longrightarrow \operatorname{Im} f \oplus \operatorname{Ker} f$$
$$v \longmapsto (f(v), v - f'f(v))$$

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "h_1"', "\wr"] & V_2 \arrow[d, "h_2 = 1_{V_2}"] \\ \operatorname{Im} f \oplus \operatorname{Ker} f \arrow[r, "{(\nu,0)}"] & V_2 \end{tikzcd} \qquad \nu : \operatorname{Im} f \hookrightarrow V_2$$

$$\begin{tikzcd} \operatorname{Im} f \arrow[r, hook, "\nu"] & V_2 & = (0) \text{ (ii)} \\ \oplus & & \text{or} \\ \operatorname{Ker} f \arrow[r, "0"] & 0 & = (0) \text{ (i)} \end{tikzcd}$$

(ii) $$\begin{tikzcd} \operatorname{Ker} f \arrow[r, "0"] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[d, equal] \\ k^t \arrow[r] & 0 \end{tikzcd}$$ (V,f) indec.

$\Downarrow$

$$\overset{\wr}{\left(k \longrightarrow 0\right)^t} \qquad t = 1. \; \underset{(V,f)}{\underset{\wr}{(k \longrightarrow 0)}}$$

$\mathrm{Im}f \hookrightarrow V_2$

Know: $V_2 = \mathrm{Im}f \oplus V_2'$

$$\begin{tikzcd}
\mathrm{Im}f \arrow[r, hook] \arrow[d, equal, "1_{\mathrm{Im}f}"'] & V_2 \arrow[d, equal] \\
\mathrm{Im}f \arrow[r, "\begin{pmatrix} 1_{\mathrm{Im}f} \\ 0 \end{pmatrix}"] & \mathrm{Im}f \oplus V_2'
\end{tikzcd}$$

$\|$

$\mathrm{Im}f \xrightarrow{1_{\mathrm{Im}f}} \mathrm{Im}f \quad = (0) \quad (i)$

$\oplus$ $\qquad$ or

$0 \xrightarrow{0} V_2' \quad = (0) \quad (ii)$

(i)
$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, equal] & V_2' \arrow[d, "\wr"] \\
0 \arrow[r, "0"] & k^t
\end{tikzcd}$$

$= (0 \longrightarrow k)^t \xRightarrow[\text{indec.}]{(V,f)} t=1 \ (V,f) \simeq (0 \xrightarrow{0} k)$

(ii)
$$\begin{tikzcd}
\mathrm{Im}f \arrow[r, "1_{\mathrm{Im}f}"] \arrow[d, "\wr \varphi"] & \mathrm{Im}f \arrow[d, "\wr \varphi"] \\
k^t \arrow[r, "1_{k^t}"] & k^t
\end{tikzcd} \simeq (k \xrightarrow{1} k)^t$$

$\xRightarrow[\text{indec.}]{(V,f)} t=1$ and $(V,f) \simeq (k \xrightarrow{1} k)$

Check: $k \xrightarrow{1} k$, $k \to 0$, $0 \to k$ are indec.

Hence: The only indec rep. are the ones above "⇒" The only indec. left $k\Gamma$-mod's are

$k\Gamma e_1$, $k\Gamma e_1 / \langle \alpha e_1 \rangle$ and $S_2 = k\Gamma e_2$

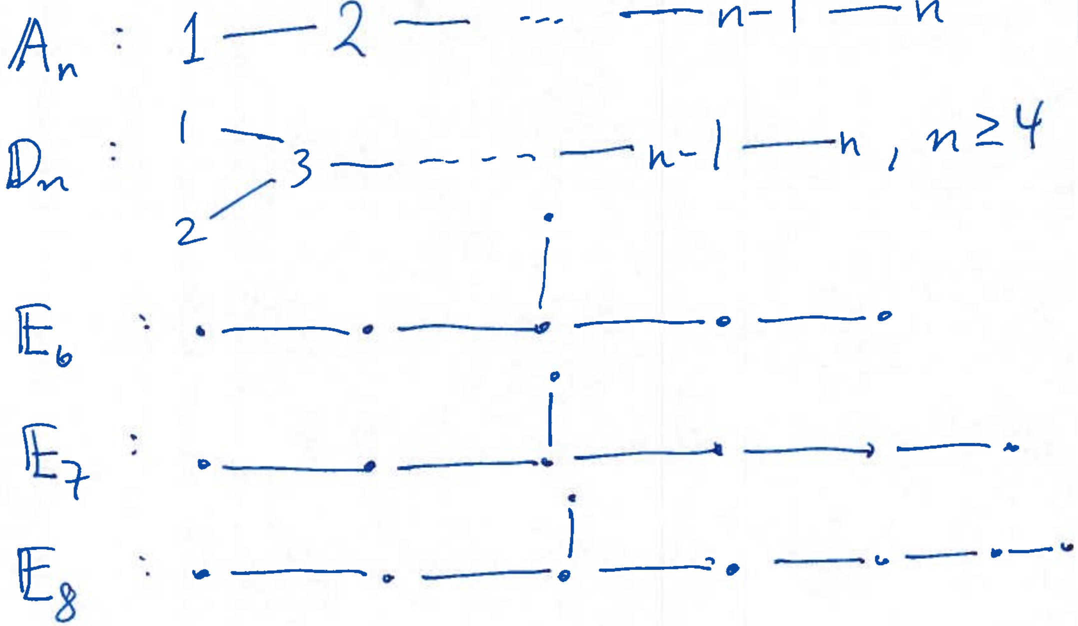
$\Rightarrow k\Gamma$ is of finite representation type.

Theorem k field, char $k = p$, G finite group, $p \mid |G|$,
kG of finite representation type
$\Longleftrightarrow \{$All p-Sylow subgroups of G are cyclic

<u>Theorem 5</u> Γ connected quiver without oriented cycles, k field.

$k\Gamma$ is of finite representation type

$\Longleftrightarrow$ $\begin{cases}\text{The underlying graph of } \Gamma \\ \text{is a Dynkin diagram}\end{cases}$



Quiver with relations

Can all algebras over a field k be represented as $k\Gamma$?

No: $\Lambda = k[x]/\langle x^2\rangle \neq k\Gamma$ for all quivers Γ.

$\dim_k \Lambda = 2$, Λ not semisimple

Assume that $\Lambda \simeq k\Gamma$.

$2 = \dim_k k\Gamma \geq \#$ vertices in Γ.

$\Gamma : 1 \quad 2 \implies k\Gamma$ is semisimple $\#$

$\implies \Gamma$ has one vertex, $1 \circlearrowright \alpha$

$\implies \dim_k k\Gamma = \infty$ $\#$

But, $\Gamma : 1 \circlearrowright \alpha$, $\Lambda \simeq k\Gamma/\langle \alpha^2\rangle$.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, k field.

DEF: (a) A relation σ in the quiver Γ over k is a k-linear combination of paths

$$\sigma = a_1 p_1 + a_2 p_2 + \cdots + a_t p_t$$

where $a_i \in k$, $e(p_i) = e(p_1)$ and $s(p_i) = s(p_1)$ for all i, and $\ell(p_i) \geq 2$ (the length of the path p_i).

(b) If $\rho = \{\sigma_\ell\}_{\ell \in T}$ is a set of relations in Γ over k, then (Γ, ρ) is a quiver with relations over k.

Example Γ :

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$

, k field

$\sigma = \gamma\alpha - \delta\beta$

$\Lambda = k\Gamma/\langle\sigma\rangle$. Let M be a left Λ-module. Any left Λ-module is a left $k\Gamma$-module, since $k\Gamma \xrightarrow{\pi} k\Gamma/\langle\sigma\rangle = \Lambda$.

$\implies M$ gives rise to a representation of Γ.

$$\begin{tikzcd} & e_1 M \arrow[dl, "f_\alpha = \alpha\cdot -"'] \arrow[dr, "\beta\cdot - = f_\beta"] & \\ e_2 \arrow[dr, "f_\gamma = \gamma\cdot -"'] & & e_3 M \arrow[dl, "\delta\cdot - = f_\delta"] \\ & e_4 M & \end{tikzcd}$$

$\sigma \in k\Gamma$, $m \in M$

$\sigma \cdot m \overset{\text{def}}{=} \pi(\sigma)\cdot m$

$= 0 \cdot m = 0$, $\forall m \in M$.

$m = e_1 m + e_2 m + e_3 m + e_4 m$

$\sigma = e_4 \sigma e_1$

$$0 = \sigma \cdot m = (\gamma\alpha - \delta\beta) m = (\gamma\alpha - \delta\beta) e_1 m \quad \forall m \in M$$

$$= \gamma(\alpha e_1 m) - \delta\cdot(\beta \cdot e_1 m) = f_\gamma f_\alpha(e_1 m) - f_\delta f_\beta(e_1 m)$$

$$= \underbrace{(f_\gamma f_\alpha - f_\delta f_\beta)}_{f_\sigma}(e_1 m) \implies f_\sigma(e_1 M) = 0$$

$$\implies f_\sigma = 0.$$

Hence Λ-module M corresponds to a representation of Γ satisfying the relation σ ($f_\sigma = 0$).

Conversely, we claim that a representation (V,f) of Γ such that $f_\sigma = f_\gamma f_\alpha - f_\delta f_\beta = 0$ gives a module over Λ.

Recall: $I \subseteq R$ ideal : R/I-module M is the same as an R-module M such that $I \cdot M = (0)$.

$$M = V(1) \oplus V(2) \oplus V(3) \oplus V(4) \leftarrow k\Gamma\text{-module}$$

$$\left[\begin{array}{l} e_1 \cdot (v_1, v_2, v_3, v_4) = (v_1, 0, 0, 0) \\ \alpha \cdot (v_1, v_2, v_3, v_4) = (0, f_\alpha(v_1), 0, 0) \end{array}\right]$$

$$\begin{aligned}\sigma(v_1, v_2, v_3, v_4) &= (\gamma\alpha - \delta\beta)(v_1, v_2, v_3, v_4) \\ &= (0, 0, 0, f_\gamma f_\alpha(v_1) - f_\delta f_\beta(v_1)) \\ &= (0, 0, 0, \underbrace{(f_\gamma f_\alpha - f_\delta f_\beta)}_{f_\sigma}(v_1)) = (0,0,0,0)\end{aligned}$$

$\Rightarrow$ M is a Λ-module ($\Lambda = k\Gamma/\langle\sigma\rangle$).

Example $\Gamma \quad 1 \circlearrowleft \alpha$, $\rho = \{\alpha^2\}$, k field.

$\Lambda = k\Gamma/\langle\alpha^2\rangle$. Find all indec. Λ-modules.

M left Λ-module $\leftrightsquigarrow$ (V,f) repr. of Γ satisfying the relation α^2, i.e. $f_{\alpha^2} = (f_\alpha)^2 = 0$.

$$V \circlearrowleft f_\alpha, \quad (f_\alpha)^2 = 0.$$

$\Rightarrow$ The minimal polynomial of f_α is x or x^2.

$\Rightarrow$ The invariant factors of f_α are x or x^2.

$\Rightarrow$ The matrix of f_α is similar to a direct sum of companion matrices of x and x^2,

$$M_{(x)} = (0) \quad \text{and} \quad M_{(x^2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let T be the matrix of f_α w.r.t. some basis B. Then ∃ an invertible matrix P such that

$$T = P \underbrace{\left(\begin{array}{c|c} r\left\{\begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix}\right. & \overbrace{\qquad 0 \qquad}^{2s} \\ \hline 0 & \begin{matrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\ 0 & & \ddots \; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{matrix} \end{array}\right)}_{T_0} P^{-1} \Rightarrow \underline{TP = PT_0}$$

$$\begin{tikzcd} V \arrow[loop right, "T"] & & V \arrow[loop right, "T_0"] \arrow[ll, leftrightarrow, "P"'] \end{tikzcd} \quad \text{isom. of repr.}$$

$$\| $$

$$(k \circlearrowleft 0)^r \oplus \left(k^2 \circlearrowleft \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)^s$$

Show: $k \circlearrowleft 0$ and $k^2 \circlearrowleft \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are indec.

$\updownarrow k\Gamma/\langle\alpha\rangle \qquad \updownarrow \Lambda = k\Gamma/\langle\alpha^2\rangle \Rightarrow$ Λ is of fin. repr. type

Finite length.

Λ ring, A a (left) Λ-module

DEF: A has finite length if there exists a finite filtration

$$\mathcal{F}: A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n \underset{\underbrace{\quad}_{A_{n+1}}}{\overset{\supseteq}{}} (0)$$

of submodules of A such that $A_i/A_{i+1} = (0)$ or simple for $i = 0, 1, \ldots, n$.

$\mathcal{F}$ is a generalized composition series of A, and if $A_i/A_{i+1} \neq (0)$ for all i, then $\mathcal{F}$ is a composition series of A. If $S = A_i/A_{i+1} \neq (0)$, then S is called a composition factor of A.

Let S be a simple Λ-module.

Let $m_S^{\mathcal{F}}(A) \overset{\text{def}}{=} |\{i \mid A_i/A_{i+1} \simeq S\}|$

and $l_{\mathcal{F}}(A) \overset{\text{def}}{=} \sum\limits_{\substack{[S] \text{ isomorphism} \\ \text{classes of simples}}} m_S^{\mathcal{F}}(A)$.

$$l(A) \overset{\text{def}}{=} \min_{\mathcal{F} \text{ gen. comp series}} l_{\mathcal{F}}(A)$$

Examples

(1) Λ ring, S simple Λ-module.

Composition series: $S \supseteq (0)$

Composition factors: $\{S\}$.

$$\Rightarrow m_T(S) = \begin{cases} 1, & \text{if } T \simeq S \\ 0, & \text{otherwise.} \end{cases} \Rightarrow l(S) = 1$$

(2) $\Lambda = k[x]$, $f(x)$ irreducible

$S_f = k[x]/\langle f(x) \rangle$ – simple Λ-module.

$\rightarrow l(S_f) = 1$, while $\dim_k S_f = \deg f(x)$.

(3) $\Gamma: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$.

$M = \Lambda e_1 \rightsquigarrow k \xrightarrow{1} k \xrightarrow{1} k$

$$\begin{tikzcd}
\mathcal{F}: & k \arrow[d, "1"'] & 0 \arrow[l] \arrow[d, "0"] & 0 \arrow[l] \arrow[d, "0"] & 0 \arrow[l] \arrow[d, "0"] \\
& k \arrow[d, "1"'] & k \arrow[l, "1"'] \arrow[d, "1"] & 0 \arrow[l] \arrow[d, "0"] & 0 \arrow[l] \arrow[d, "0"] \\
& k \arrow[d, phantom, "\parallel"] & k \arrow[l, "1"'] \arrow[d, phantom, "\parallel"] & k \arrow[l, "1"'] \arrow[d, phantom, "\parallel"] & 0 \arrow[l] \arrow[d, phantom, "\parallel"] \\
& V_0 \arrow[d, squiggly] & V_1 \arrow[d, squiggly] & V_2 \arrow[d, leftrightarrow] & V_3 \\
& M = M_0 \arrow[r, phantom, "\supseteq"] & M_1 \arrow[r, phantom, "\supseteq"] & M_2 \arrow[r, phantom, "\supseteq"] & M_3 = (0)
\end{tikzcd}$$

$$V_0/V_1 \simeq \begin{matrix} k \\ \downarrow \\ 0 \\ \downarrow \\ 0 \end{matrix} \rightsquigarrow S_1 \qquad V_1/V_2 \simeq \begin{matrix} 0 \\ \downarrow \\ k \\ \downarrow \\ 0 \end{matrix} \rightsquigarrow S_2$$

$$V_2/V_3 = \begin{matrix} 0 \\ \downarrow \\ 0 \\ \downarrow \\ k \end{matrix} \rightsquigarrow S_3$$

$\Rightarrow l_{\mathcal{F}}(M) = 3$

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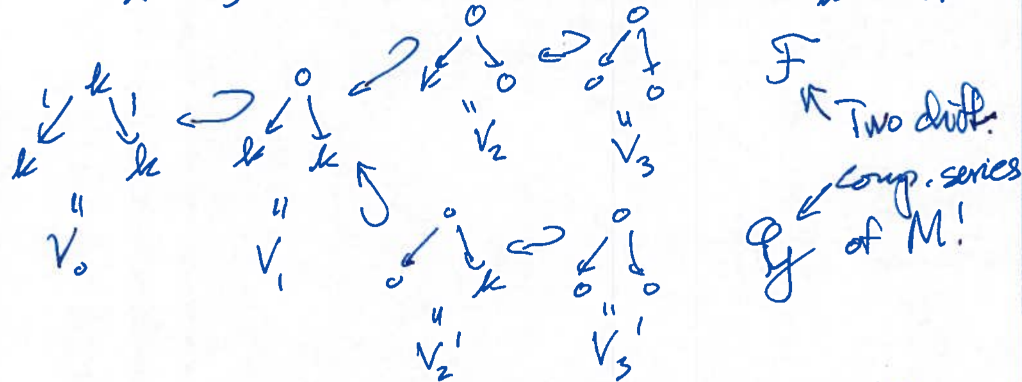
(4) Γ $\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 & & 3 \end{tikzcd}$, k field, $M = k\Gamma e_1$ $\rightsquigarrow$ $\begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\ k & & k \end{tikzcd}$



$$\begin{tikzcd} V_0/V_1 \arrow[d, leftrightsquigarrow] & V_1/V_2 \arrow[d, leftrightsquigarrow] & V_2/V_3 \arrow[d, leftrightsquigarrow] & V_0/V_1 \arrow[d, leftrightsquigarrow] & V_1/V_2' \arrow[d, leftrightsquigarrow] & V_2'/V_3' \arrow[d, leftrightsquigarrow] \\ S_1 & S_3 & S_2 & S_1 & S_2 & S_3 \end{tikzcd}$$

<u>Note</u>: (1) Composition series are not unique!

(2) $\ell_{\mathcal{F}}(M) = \ell_{\mathcal{G}}(M)$

(3) The set of ~~composition~~ factors is the same for $\mathcal{F}$ ~~$\mathcal{G}$~~ and $\mathcal{G}$.

The ~~proof~~ of Jordan-Hölder theorem goes by induction on length and using short exact sequences.

DEF: $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a (short) ~~exact sequence~~ of (left) Λ-modules if

(i) f is injective (1-1),

(ii) g is surjective (onto),

(iii) $\operatorname{Im} f = \operatorname{Ker} g$.

<u>Note</u>: (1) $A \subseteq B$ Λ-modules, submodule

Then

$$0 \to A \hookrightarrow B \to B/A \to 0$$

is an exact sequence.

(2) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence, then

(a) $C = \operatorname{Im} g \simeq B/\operatorname{Ker} g = B/\operatorname{Im} f$,
$\operatorname{Im} f \simeq A$.

(b) $B = (0) \Rightarrow A = (0)$ and $C = (0)$.

<u>Examples</u>

(1) $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$
exact.

(2) M, N Λ-modules

$$0 \to M \xrightarrow{\binom{1}{0}} M \oplus N \xrightarrow{(0\ 1)} N \to 0$$
$$m \longmapsto (m, 0)$$
$$(m, n) \longmapsto n$$
exact.

(3) $\Lambda = k\Gamma/\langle\rho\rangle$, $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact seq. of Λ-mod.

$$\begin{tikzcd}
 & V_A(i) \arrow[r, "f|_{V_A(i)}"] \arrow[d, equal] & V_B(i) \arrow[r] \arrow[d, equal] & V_C(i) \arrow[d, equal] & \\
0 \arrow[r] & e_iA \arrow[r, "f|_{e_iA}"] & e_iB \arrow[r, "g|_{e_iB}"] & e_iC \arrow[r] & 0
\end{tikzcd}$$

$\left\{\begin{array}{l}\text{exact}\\ \text{seq. for}\\ \text{all } i.\end{array}\right.$

Hence, $0 \to (V', f') \xrightarrow{g} (V, f) \xrightarrow{h} (V'', f'') \to 0$

is <u>an exact sequence of rep's</u> if

$$0 \longrightarrow V'(i) \xrightarrow{g(i)} V(i) \xrightarrow{h(i)} V''(i) \longrightarrow 0$$

is exact, for all $i \in \Gamma_0$.

<u>Exercise</u>: $f: A \to B$ and $g: B \to C$, Λ-hom, $B' \subseteq B$ submodule.

(1) $f^{-1}(B') = \{a \in A \mid f(a) \in B'\} \subseteq A$ submodule

(2) $g(B') = \{g(b') \mid b' \in B'\} \subseteq C$ submodule.

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence and let $\mathcal{F}$ be a generalized composition series of B.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r] \arrow[d, equal] & 0 \\
 & A_0 = f^{-1}(B_0) \arrow[d, phantom, "\cup|"] & B_0 \arrow[d, phantom, "\cup|"] & g(B_0) = C_0 \arrow[d, phantom, "\cup|"] & \\
 & A_1 = f^{-1}(B_1) \arrow[d, phantom, "\cup|"] & B_1 \arrow[d, phantom, "\cup|"] & g(B_1) = C_1 \arrow[d, phantom, "\cup|"] & \\
 & A_2 = f^{-1}(B_2) \arrow[d, phantom, "\cup|"] & B_2 \arrow[d, phantom, "\cup|"] & g(B_2) = C_2 \arrow[d, phantom, "\cup|"] & \\
 & \vdots \arrow[d, phantom, "\cup|"] & \vdots \arrow[d, phantom, "\cup|"] & \vdots \arrow[d, phantom, "\cup|"] & \\
 & A_n = f^{-1}(B_n) = (0) & B_n = (0) & g(B_n) = C_n = (0). & \\
 & \mathcal{F}' \quad f \text{ 1-1.} \arrow[u] & \mathcal{F} & \mathcal{F}'' &
\end{tikzcd}$$

<u>Proposition 7</u>

(a) $\mathcal{F}'$ is a generalized comp. series of A.

$\mathcal{F}''$ ——— " ——— C.

(b) $m_S^{\mathcal{F}}(B) = m_S^{\mathcal{F}'}(A) + m_S^{\mathcal{F}''}(C)$.

$\forall S$ simple.

Proof: (a) (I) Claim:

(*) $0 \longrightarrow A_i \xrightarrow{f|_{A_i}} B_i \xrightarrow{g|_{B_i}} C_i \longrightarrow 0$ exact. $\forall i$.

By definition: $f(A_i) \subseteq B_i$ and $g(B_i) = C_i$.

- $f|_{A_i} : A_i \longrightarrow B_i$ is 1-1, since f is 1-1.
- $g|_{B_i} : B_i \longrightarrow C_i$ is onto by def. of C_i
- Since $f(A_i) \subseteq B_i$ and $g \cdot f = 0$, then $g|_{B_i} \cdot f|_{A_i} = 0. \Rightarrow \operatorname{Im} f|_{A_i} \subseteq \operatorname{Ker} g|_{B_i}$

Let $b \in \operatorname{Ker} g|_{B_i} \Rightarrow b \in \operatorname{Ker} g = \operatorname{Im} f$. So $\exists\, a \in A$ such that $f(a) = b$, i.e.

$a \in f^{-1}(B_i) = A_i$

$\Longrightarrow \operatorname{Ker} g|_{B_i} \subseteq \operatorname{Im} f|_{A_i} \Rightarrow \operatorname{Ker} g|_{B_i} = \operatorname{Im} f|_{A_i}$

$\Longrightarrow$ (*) is exact.

(II) Claim: The following diagram is exact and commutative:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "f"] \arrow[d, hook] & B_{i+1} \arrow[r, "g"] \arrow[d, hook] & C_{i+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & A_i \arrow[r, "f"] \arrow[d, "p_i'"] & B_i \arrow[r, "g"] \arrow[d, "p_i"] & C_i \arrow[r] \arrow[d, "p_i''"] & 0 \\
0 \arrow[r] & A_i/A_{i+1} \arrow[r, "\bar f"] \arrow[d] & B_i/B_{i+1} \arrow[r, "\bar g"] \arrow[d] & C_i/C_{i+1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\eta_i : a_i \mapsto a_i + A_{i+1}$

where $\bar f(a_i + A_{i+1}) = f(a_i) + B_{i+1}$ and

$\bar g(b_i + B_{i+1}) = g(b_i) + C_{i+1}$.

Easy to see that the diagram is commutative, given that everything is well-defined.

(i) $\bar f$ well-defined: Assume that $a_i + A_{i+1} = a_i' + A_{i+1}$

i.e. $a_i - a_i' \in A_{i+1}$

$$\left.\begin{array}{l} \Longrightarrow f(a_i - a_i') \in B_{i+1} \\ \qquad f(a_i) - f(a_i') \end{array}\right\} \Rightarrow \begin{array}{l} \bar f(a_i + A_{i+1}) = f(a_i) + B_{i+1} \\ \qquad\qquad\qquad\quad \| \\ \bar f(a_i' + A_{i+1}) = f(a_i') + B_{i+1} \end{array}$$

$\Rightarrow \bar f$ is well-defined.

(ii) $\bar g$ well-defined: similar.

(iii) $\underline{\eta_i : \text{exact}}$:

(1) $\underline{\bar{f}\ 1\text{-}1}$: Assume that $\bar{f}(a_i+A_{i+1})=0\ (a_i\in A_i)$

$$\overset{\shortparallel}{f(a_i)+B_{i+1}} \rightarrow f(a_i)\in B_{i+1}$$

$\Longrightarrow a_i \in A_{i+1}$ by def. $\Rightarrow a_i+A_{i+1}=0 \rightarrow \bar{f}$ 1-1.

(2) $\underline{\bar{g}\text{ onto}}$: g, p_i'' onto $\Rightarrow p_i''g=\bar{g}p_i$ onto $\Rightarrow \bar{g}$ onto

(3) $\underline{\text{Im}\bar{f}=\text{Ker}\bar{g}}$:

- $\bar{g}\bar{f}(a_i+A_{i+1})=g(f(a_i)+B_{i+1})=\overset{0}{\overset{\shortparallel}{gf(a_i)}}+C_{i+1}$ $=0$

$\Longrightarrow \text{Im}\bar{f}\subseteq \text{Ker}\bar{g}$.

- Let $b_i+B_{i+1}\in\text{Ker}\bar{g}$, $b_i\in B_i$, i.e.

$$0=\bar{g}(b_i+B_{i+1})=g(b_i)+C_{i+1},$$

hence $g(b_i)\in C_{i+1}$. Choose $b_{i+1}\in B_{i+1}$ such that $g(b_{i+1})=g(b_i)$. Then $b_i-b_{i+1}\in B_i$, since $b_i, b_{i+1}\in B_i$.

$\Rightarrow g(b_i-b_{i+1})=0$

$\Rightarrow b_i-b_{i+1}\in \text{Ker}g=\text{Im}f$

Choose $a_i\in A_i$ such that $f(a_i)=\underbrace{b_i-b_{i+1}}_{\in B_i}$

$\Rightarrow \bar{f}(a_i+A_{i+1})=f(a_i)+B_{i+1}$

$$=b_i-\underbrace{b_{i+1}}_{\in B_{i+1}}+B_{i+1}=b_i+B_{i+1}$$

$\Longrightarrow \text{Ker}\bar{g}\subseteq \text{Im}\bar{f}$

$\Longrightarrow \text{Ker}\bar{g}=\text{Im}\bar{f} \Longrightarrow \eta_i$ exact.

$\underline{\text{Have}}$: $B_i/B_{i+1}=(0)\Longrightarrow A_i/A_{i+1}=(0)$ and $C_i/C_{i+1}=(0)$

$\underset{\substack{\text{12}\\ S}}{B_i/B_{i+1}}$ simple $\Longrightarrow A_i/A_{i+1}=(0)$ or $A_i/A_{i+1}\simeq S$.

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(0)=\text{Im}\bar{f}=\text{Ker}\bar{g} \qquad \underset{\text{Ker}\bar{g}}{\overset{\shortparallel}{\text{Im}\bar{f}}}=B_i/B_{i+1} \Rightarrow \bar{g}=0.$$

$$S\simeq B_i/B_{i+1}\simeq C_i/C_{i+1} \qquad \Downarrow$$

$$C_i/C_{i+1}=\underset{(0).}{\overset{}{\underset{\shortparallel}{\text{Im}\bar{g}}}}$$

i.e. $A_i/A_{i+1}=(0)$ and $C_i/C_{i+1}\simeq S$.

or

$A_i/A_{i+1}\simeq S$ and $C_i/C_{i+1}=(0)$

$\Rightarrow \underset{\mathcal{F}''}{\mathcal{F}'}$ is a $\underline{\text{generalized composition series}}$ of $\underset{C.}{A}$ $\underset{\text{11}}{}$

(b) Immediate consequence of the proof of (a). □

Corollary 8

Given A, B and C Λ-modules with B of finite length and an exact

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then $\ell(A) + \ell(C) \leq \ell(B)$

In particular, A and C have finite length.

Proof: Let $\mathcal{F}$ be a generalized comp. series of B such that $\ell(B) = \ell_{\mathcal{F}}(B)$. Keeping the notation from before, we get

$$\ell_{\mathcal{F}'}(A) + \ell_{\mathcal{F}''}(C) = \ell_{\mathcal{F}}(B) = \ell(B)$$

Since $\ell_{\mathcal{F}'}(A) \geq \ell(A)$ and $\ell_{\mathcal{F}''}(C) \geq \ell(C)$, we obtain

$$\ell(A) + \ell(C) \leq \ell(B). \quad \square$$

Theorem 9 (Jordan-Hölder)

Let B be a Λ-module of finite length, $\mathcal{F}$ and $\mathcal{G}$ gen. comp. series of B.

Then

(a) $m_S^{\mathcal{F}}(B) = m_S^{\mathcal{G}}(B) \left(\overset{\text{def}}{=} m_S(B)\right)$

(b) $\ell_{\mathcal{F}}(B) = \ell_{\mathcal{G}}(B) \left(\overset{\text{def}}{=} \ell(B)\right)$.

Proof: (a) Induction on $\ell(B)$.

① $\underline{\ell(B) = 1}$: Then $B = S$ - simple and $m_S^{\mathcal{F}}(B) = 1 \; \forall \; \mathcal{F}$ and $m_{S'}^{\mathcal{F}}(B) = 0$ for all simple modules $S' \neq S$. Hence, (a) holds.

② Assume that $n = \ell(B) > 1$, and assume that (a) is shown for C with $\ell(C) < n$. Since B is not simple, choose $A \subseteq B$ with $(0) \subsetneq A \subsetneq B$. Have an exact sequence

$$0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$$

Then A and B/A have finite length, by Corollary 8 and $\ell(A) + \ell(B/A) \leq \ell(B)$,

$A \neq (0) \Rightarrow \ell(A) > 0$, $B/A \neq (0) \Rightarrow \ell(B/A) > 0$

Let $\mathcal{F}'$ and $\mathcal{G}'$, and $\mathcal{F}''$ and $\mathcal{G}''$ be the induced gen. comp. series for A and B/A respectively. Then

$$\begin{array}{ccccc} m_S^{\mathcal{F}}(B) & = & m_S^{\mathcal{F}'}(A) & + & m_S^{\mathcal{F}''}(B/A) \\ & \text{Prop. 7} & \| & \substack{\text{By induction}\\ \ell(A)<\ell(B)\\ \ell(B/A)<\ell(B)} & \| \\ m_S^{\mathcal{G}}(B) & = & m_S^{\mathcal{G}'}(A) & = & m_S^{\mathcal{G}''}(B/A) \end{array}$$

$\Longrightarrow m_S^{\mathcal{F}}(B) = m_S^{\mathcal{G}}(B) \quad \forall S$ simple

(b) Follows directly from (a), $\ell_{\mathcal{F}}(B) = \ell_{\mathcal{G}}(B) \overset{\text{def}}{=} \ell(B)$ □

<u>Note</u> (1) It follows that if B has finite length, then the set of composition factors is uniquely determined up to isomorphism and multiplicity.

(2) $B \simeq C \Rightarrow \ell(B) = \ell(C)$.

<u>Proposition 10</u>

If $0 \to A \to B \to C \to 0$ is exact and $\ell(B) < \infty$, then

$$\ell(B) = \ell(A) + \ell(C)$$

<u>Proof:</u> Let $\mathcal{F}$ be a gen. comp. series of B. Then

$$\ell(B) = \ell_{\mathcal{F}}(B) \overset{\text{def}}{=} \sum_{\substack{[S]-\text{iso}\\ \text{classes}\\ \text{of simples}}} m_S^{\mathcal{F}}(B)$$

$$= \sum_{[S]} \left(m_S^{\mathcal{F}'}(A) + m_S^{\mathcal{F}''}(C) \right)$$

$$= \underset{\|}{\ell_{\mathcal{F}'}(A)} + \underset{\|}{\ell_{\mathcal{F}''}(C)}$$

$$= \ell(A) + \ell(C).$$ □

<u>Examples</u> (1) $\Lambda = k[x]/\langle x^2 \rangle \rightsquigarrow \Gamma : 1 \circlearrowright \alpha$, $\Lambda \simeq k\Gamma/\langle \alpha^2 \rangle$

$$M = \Lambda \rightsquigarrow k^2 \circlearrowright \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$M = M_0 = k^2 \circlearrowright \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xhookleftarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \underset{\overset{\|}{M_1}}{k \circlearrowright 0} \hookleftarrow \underset{\overset{\|}{M_2}}{0 \circlearrowright 0}$$

$M_0/M_1 \simeq k \circlearrowright 0$ Composition factors: $\{S, S\}$

$M_1/M_2 \simeq k \circlearrowright 0$ Multiplicity of $S = 2$

(2) Γ
$$\begin{tikzcd} & 1 \arrow[dl,"\alpha"'] \arrow[dr,"\beta"] & \\ 2 \arrow[dr,"\gamma"'] & & 3 \arrow[dl,"\delta"] \\ & 4 & \end{tikzcd}$$
$\rho = \{\gamma\alpha - \delta\beta\}$, $\Lambda = k\Gamma/\langle\rho\rangle$

Let $M = \Lambda e_1$. Find $\ell(M)$.

$$\begin{tikzcd} & k \arrow[dl,"1"'] \arrow[dr,"1"] & & & e_1M \arrow[dl,"\alpha\cdot -"'] \arrow[dr,"\beta\cdot -"] & & & k\bar{e}_1 \arrow[dl,"\alpha\cdot -"'] \arrow[dr,"\beta\cdot -"] & \\ k \arrow[dr,"1"'] & & k \arrow[dl,"1"] & e_2M \arrow[dr,"\gamma\cdot -"'] & & e_3M \arrow[dl,"\delta\cdot -"] & k\bar{\alpha} \arrow[dr,"\gamma\cdot -"'] & & k\bar{\beta} \arrow[dl,"\delta\cdot -"] \\ & k & & & e_4M & & & k\overline{\gamma\alpha}=k\overline{\delta\beta} & \end{tikzcd}$$

$$\begin{tikzcd} & k \arrow[dl,"1"'] \arrow[dr,"1"] & & & 0 \arrow[dl] \arrow[dr] & & & 0 \arrow[dl] \arrow[dr] & & & 0 \arrow[dl] \arrow[dr] & & & 0 \arrow[dl] \arrow[dr] & \\ k \arrow[dr,"1"'] & & k \arrow[dl,"1"] & k \arrow[l, hook] \arrow[dr,"1"'] & & k \arrow[dl,"1"] & 0 \arrow[l, hook] \arrow[dr] & & k \arrow[dl,"1"] & 0 \arrow[l, hook] \arrow[dr] & & 0 \arrow[dl] & 0 \arrow[l, hook] \arrow[dr] & & 0 \arrow[dl] \\ & k & & & k & & & k & & & k & & & 0 & \\ & M=M_0 \arrow[u, equal] & & & M_1 \arrow[u, equal] & & & M_2 \arrow[u, equal] & & & M_3 \arrow[u, equal] & & & M_4 \arrow[u, equal] & \end{tikzcd}$$

$$\begin{tikzcd} & & k \arrow[dl] \arrow[dr] & & & & 0 \arrow[dl] \arrow[dr] & & & & 0 \arrow[dl] \arrow[dr] & & \\ M_0/M_1 \cong & 0 \arrow[dr] & & 0 \arrow[dl] & M_1/M_2 \cong & k \arrow[dr] & & 0 \arrow[dl] & M_2/M_3 \cong & 0 \arrow[dr] & & k \arrow[dl] & M_3/M_4 \\ & & 0 \arrow[d, squiggly] & & & & 0 \arrow[d, squiggly] & & & & 0 \arrow[d, squiggly] & & M_3 \arrow[u, phantom, sloped, "\cong"] \arrow[d, squiggly] \\ & & S_1 & & & & S_2 & & & & S_3 & & S_4 \end{tikzcd}$$

$\Rightarrow \ell(M) = 4$, comp. factors: $\{S_1, S_2, S_3, S_4\}$.

<u>Note</u>: M has comp. factors $\{S_1, S_2, S_3, S_4\}$

The module $N = S_1 \oplus S_2 \oplus S_3 \oplus S_4$ has the same comp. factors:

$$\underset{\underset{N_0}{\parallel}}{N = S_1 \oplus S_2 \oplus S_3 \oplus S_4} \supseteq \underset{\underset{N_1}{\parallel}}{(0) \oplus S_2 \oplus S_3 \oplus S_4} \supseteq \underset{\underset{N_2}{\parallel}}{(0) \oplus (0) \oplus S_3 \oplus S_4}$$
$$\cup\!\!\!\mid$$
$$N_3 = (0) \oplus (0) \oplus (0) \oplus S_4$$
$$\cup\!\!\!\mid$$
$$N_4 = (0).$$

Have $N_i/N_{i+1} \cong S_{i+1}$, $i = 0,1,2,3$.

So in general, a module is not uniquely determined by its composition factors.

<u>Proposition 11</u>: Given a Λ-module A of finite length and a Λ-homomorphism $f: A \to A$. The following are equivalent

(a) f is an isomorphism

(b) f is a monomorphism (1-1).

(c) f is an epimorphism (onto).

<u>Proof:</u> Clearly, (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) by def.
We have the exact sequence

$$0 \longrightarrow f(A) \hookrightarrow A \longrightarrow A/f(A) \longrightarrow 0$$

<u>(b) $\Rightarrow$ (c):</u> f 1-1 $\Rightarrow A \simeq f(A) \Rightarrow \ell(A) = \ell(f(A))$

$$\Rightarrow \ell(A/f(A)) = 0 \Rightarrow f(A) = A$$

$$\Rightarrow f \text{ onto}$$

<u>(c) $\Rightarrow$ (a):</u> f onto $\Rightarrow f(A) = A \Rightarrow \ell(A/f(A)) = 0$

$$\Rightarrow \ell(A) = \ell(f(A))$$

$$0 \longrightarrow \operatorname{Ker} f \hookrightarrow A \longrightarrow f(A) \longrightarrow 0 \text{ exact}$$

$\Rightarrow \ell(\operatorname{Ker} f) = 0 \Rightarrow \operatorname{Ker} f = (0) \Rightarrow f$ 1-1

$\Rightarrow f$ isom. □

<u>Note:</u> The proof of Proposition 11 holds for all $f: A \to B$ with $\ell(A) = \ell(B)$.
So if $\ell(A) = \ell(B)$ and $f: A \to B$, then
f isom $\Leftrightarrow f$ 1-1 $\Leftrightarrow f$ onto.

Recall: $0 \to A \longrightarrow B \longrightarrow C \to 0$ exact and $\ell(B) < \infty$, then $\ell(A)$ and $\ell(C)$ are finite too.

<u>Proposition 12</u> If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact and A and C have finite length, then also B has finite length and

$$\ell(B) = \ell(A) + \ell(C).$$

<u>Proof:</u> Let

$$\mathcal{F}' : A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n = (0)$$

and $\mathcal{F}'' : C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{m-1} \supseteq C_m = (0)$

be two comp. series of A and C, respectively.
Consider the following chain of submods of B:

$$\mathcal{F} : B = g^{-1}(C) \supseteq g^{-1}(C_1) \supseteq g^{-1}(C_2) \supseteq \cdots \supseteq g^{-1}(C_m) = \operatorname{Ker} g.$$

$$\|$$

$$(0) = f(A_n) \subseteq f(A_{n-1}) \cdots \subseteq f(A_2) \subseteq f(A_1) \subseteq f(A) = \operatorname{Im} f$$

We want to show that $\mathcal{F}$ is a comp. series of B.
Let $g_i = g|_{g^{-1}(C_i)} : g^{-1}(C_i) \longrightarrow C_i \quad (b_i \mapsto g(b_i))$
Then g_i is clearly surjective (since g is)
The composition $\psi_i : \pi_i g_i : g^{-1}(C_i) \to C_i \xrightarrow{\pi_i} C_i/C_{i+1}$

$$c_i \mapsto c_i + C_{i+1}$$

is onto (comp. of two onto maps) and we have $b_i \in \operatorname{Ker} \psi_i \Leftrightarrow \pi_i(g_i(b_i)) = 0$ in C_i/C_{i+1}

$$\Leftrightarrow g(b_i) + C_{i+1} = 0$$

$g(b_i) \in C_{i+1}$

$\iff b_i \in g^{-1}(C_{i+1})$

$\Rightarrow \operatorname{Ker}\psi_i = g^{-1}(C_{i+1})$

$\Rightarrow \underbrace{g^{-1}(C_i)/g^{-1}(C_{i+1})}_{\operatorname{Ker}\psi_i} \simeq \underbrace{C_i/C_{i+1}}_{\text{simple by def of } \mathcal{F}''} = \operatorname{Im}\psi_i$

Let $f_i = f|_{A_i} : A_i \longrightarrow f(A_i)$ (i.e. $a_i \mapsto f(a_i)$) which clearly is onto. The composition

$$\theta_i = p_i f_i : A_i \xrightarrow{f_i} f(A_i) \xrightarrow{p_i} f(A_i)/f(A_{i+1})$$
$$b_i \longmapsto b_i + f(A_{i+1})$$

is onto (comp. of two onto maps) and we have

$a_i \in \operatorname{Ker}\theta_i \iff p_i f_i(a_i) = 0$ in $f(A_i)/f(A_{i+1})$

$\iff f(a_i) + f(A_{i+1}) = 0$

$\iff f(a_i) \in f(A_{i+1})$

$\iff \exists\, a_{i+1} \in A_{i+1}$ such that $f(a_{i+1}) = f(a_i)$

$\overset{f\ 1\text{-}1}{\iff} a_i = a_{i+1}$

$\iff a_i \in A_{i+1}$

$\Rightarrow \operatorname{Ker}\theta_i = A_{i+1}$

$\Rightarrow A_i/\operatorname{Ker}\theta_i \simeq \operatorname{Im}\theta_i = f(A_i)/f(A_{i+1})$

$\qquad \| $

A_i/A_{i+1} — simple by def of $\mathcal{F}'$.

$\Rightarrow \mathcal{F}$ is a comp. series of B and

$$\ell(B) = \ell(A) + \ell(C). \qquad \square.$$

<u>DEF:</u> A collection $\mathcal{C}$ of modules (a full subcategory) is <u>closed under extensions</u> if for each exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with $A, C \in \mathcal{C}$, then B is also in $\mathcal{C}$.

Let $f\ell(\Lambda)$ be the collection of Λ-modules of finite length.

<u>Proposition 13</u>

(a) $f\ell(\Lambda)$ is closed under extensions and contains the simples. Furthermore, $f\ell(\Lambda)$ is closed under submodules and factor modules.

(b) Let $\mathcal{C}$ be a collection of Λ-modules that is closed under extensions and contains the simple Λ-modules. Then $f\ell(\Lambda) \subseteq \mathcal{C}$.

Proof: (a) Prop 12 & Corollary 8.

(b) Let $B \in fl(\Lambda)$ with $l(B) = n$. Induction on n.

n=1: Then B is simple and $B \in \mathcal{C}$.

n>1: Choose $0 \subsetneq A \subsetneq B$ submodule (possible since B is not simple).

$\Rightarrow 0 \to A \hookrightarrow B \to B/A \to 0$ exact with $l(A), l(B/A) < l(B) = n$

Induction $\Rightarrow A, B/A \in \mathcal{C}$

$\mathcal{C}$ closed under extensions $\to B \in \mathcal{C}$

$\Rightarrow fl(\Lambda) \subseteq \mathcal{C}$. □

Recall: A module M is Noetherian (Artinian) if for every ascending (descending) chain of submodules of M:

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots \subseteq M$$
$$(M \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots)$$

$\exists n$ such that

$$M_n = M_{n+1} = \cdots .$$

- M is Noetherian (Artinian) $\Leftrightarrow$ every non-empty set of submodules of M has a maximal (minimal) element.

Proposition 14 A Λ-module

$l(A) < \infty \Leftrightarrow A$ is Artinian and Noetherian.

Proof: $\Rightarrow$: WTS: $fl(\Lambda) \subseteq art(\Lambda)$ – collection of Artinian Λ-modules

Will use Prop 13 (b).

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact

Claim: $A, C \in art(\Lambda) \Rightarrow B \in art(\Lambda)$.

Proof: Consider a descending chain in $B = B_0 \supseteq B_1 \supseteq \cdots$

As before we get induced chains of submodules in A and C

$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_i \supseteq \cdots$, with $A_i := f^{-1}(B_i)$

$C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_i \supseteq \cdots$, — " — $C_i := g(B_i)$.

and induced exact sequences

$$0 \to A_i/A_{i+1} \xrightarrow{\bar{f}} B_i/B_{i+1} \xrightarrow{\bar{g}} C_i/C_{i+1} \to 0$$

Since $A, C \in \mathrm{art}(\Lambda)$, $\exists N$ such that

$A_i = A_{i+1}$ and $C_i = C_{i+1}$ for $i \geq N$.

$\Rightarrow A_i/A_{i+1} = (0) = C_i/C_{i+1}$ for $i \geq N$.

$\Rightarrow B_i/B_{i+1} = (0) \Rightarrow B_i = B_{i+1}$ for $i \geq N$.

$\Rightarrow B \in \mathrm{art}(\Lambda)$.

$\Rightarrow \mathrm{art}(\Lambda)$ is closed under extensions

Clear that $\mathrm{art}(\Lambda)$ contains the simple Λ-modules: Prop 13 (b) $\Rightarrow \mathrm{fl}(\Lambda) \subseteq \mathrm{art}(\Lambda))$

Exercise: Similarly, $\mathrm{fl}(\Lambda) \subseteq \mathrm{noeth}(\Lambda)$ – collection of noetherian Λ-modules.

$\Rightarrow \mathrm{fl}(\Lambda) \subseteq \mathrm{art}(\Lambda) \cap \mathrm{noeth}(\Lambda)$.

$\Leftarrow$: Assume that $B \neq (0)$ is Artinian and Noetherian. Since B is Artinian, B has a simple submodule $S \subseteq B$. ($\mathcal{F} = \{U \subseteq B \mid U \neq (0)\}$ – has minimal elt.)

Consider $\mathcal{F}' = \{U \subseteq B \mid \ell(U) < \infty\}$. Then $\mathcal{F}' \neq \emptyset$, since $S \in \mathcal{F}'$. Since B is Noetherian, $\mathcal{F}'$ has a maximal element $A \subseteq B$ and $A \in \mathrm{fl}(\Lambda)$.

Assume that $A \subsetneq B$, i.e. $B/A \neq (0)$.

B Artinian $\Rightarrow B/A$ Artinian.

$\Rightarrow \exists\, T \subseteq B/A$ simple submodule

Consider the natural projection $p: B \to B/A$.

Then $p|_{p^{-1}(T)}: p^{-1}(T) \longrightarrow T \; (\subseteq B/A)$

is onto and $\mathrm{Ker}\, p|_{p^{-1}(T)} = A$. Hence

$(p^{-1}(T))/A \simeq T$ and we have an exact seq.

$$0 \to \underset{\mathrm{fl}(\Lambda)}{\underset{\cap}{A}} \longrightarrow p^{-1}(T) \longrightarrow \underset{\mathrm{fl}(\Lambda)}{\underset{\cap}{T}} \to 0$$

$\Leftarrow \Rightarrow p^{-1}(T) \subseteq B$ and it is in $\mathrm{fl}(\Lambda)$

$\ell(p^{-1}(T)) = \ell(A) + 1$ ⨳

$\Longrightarrow A = B$ and $B \in \mathrm{fl}(\Lambda)$.

Note: (1) Λ ring (with 1)

Λ left Artinian $\Longleftrightarrow \ell({}_\Lambda\Lambda) < \infty$.

Proof: $\Rightarrow$: Λ left Artinian $\to \Lambda$ left Noetherian

$\Longrightarrow {}_\Lambda\Lambda \in \operatorname{art}(\Lambda) \cap \operatorname{noeth}(\Lambda)$

$\Longrightarrow \ell({}_\Lambda\Lambda) < \infty$.

$\Leftarrow$: $\ell({}_\Lambda\Lambda) < \infty \Longrightarrow {}_\Lambda\Lambda \in \operatorname{art}(\Lambda) \cap \operatorname{noeth}(\Lambda)$

$\Longrightarrow \Lambda$ left Artinian. □

(2) Λ left Artinian.

Challenge: $\operatorname{fl}(\Lambda) = \operatorname{mod}\Lambda$ - finitely generated Λ-modules.

(3) $\Lambda = \mathbb{Z}$, $M = \mathbb{Z}/(n)$.

$n = p_1^{m_1} p_2^{m_2} \cdots p_t^{m_t}$, p_i diff. primes $m_i \geq 1$.

$$m_{\mathbb{Z}/p}(M) = \begin{cases} m_i, & \text{if } p = p_i \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 15

Λ ring, B semisimple Λ-module.

TFAE:

(a) $\ell(B) < \infty$

(b) B is Artinian.

(c) B is Noetherian

Proof: Exercise. □

Radical

DEF: Λ ring. The (left) radical of Λ is the left ideal

$$\underline{r} = \operatorname{rad}\Lambda = \bigcap_{\substack{\underline{m} \text{ maximal} \\ \text{left ideals}}} \underline{m}$$

(Also called the Jacobson radical of Λ)

Know: $\underline{r}$ is a left ideal

Show: $\underline{r}$ is an ideal.

Examples

(1) If Λ is a division ring, then $\underline{r} = (0)$.

(2) $\Lambda = \mathbb{Z}$, (p) - max. ideal if p is prime

$$\underline{r} = \bigcap_{p \text{ prime}} (p) = (n) = (0).$$

$(n) \subseteq (p)$ $\forall p \Rightarrow p \mid n$ $\forall p$ prime

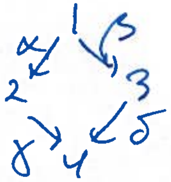
$\Rightarrow n = 0.$

(3) $\Lambda = \mathbb{Q} \times \mathbb{Q}$, $\left.\begin{array}{l} \underline{m}_1 = \mathbb{Q} \times (0) \\ \underline{m}_2 = (0) \times \mathbb{Q} \end{array}\right\}$ max ideals.

$(0) = \underline{m}_1 \cap \underline{m}_2 \supseteq \underline{r} \Rightarrow \underline{r} = (0)$.

In general, if we can find a finite set of maximal left ideals $\{\underline{m}_i\}_{i=1}^{t}$ such that $\bigcap_{i=1}^{t} \underline{m}_i = (0)$, then $\underline{r} = (0)$.

Exercise: Λ semisimple $\Rightarrow \operatorname{rad}\Lambda = (0)$.

(4) Γ 

$\rho = \{\gamma\alpha - \delta\beta\}$, k field, $\Lambda = k\Gamma/\langle\rho\rangle$.

$\operatorname{rad}\Lambda = ?$

Know: $1_\Lambda = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4$ $\quad \bar{e}_i \cdot \bar{e}_j = \begin{cases} \bar{e}_i, & i=j \\ 0 & i \neq j \end{cases}$

Exercise: ${}_\Lambda\Lambda = \Lambda\bar{e}_1 \oplus \Lambda\bar{e}_2 \oplus \Lambda\bar{e}_3 \oplus \Lambda\bar{e}_4$

$\underline{m}_1 = \Lambda\{\alpha, \beta\} \oplus \Lambda\bar{e}_2 \oplus \Lambda\bar{e}_3 \oplus \Lambda\bar{e}_4$

$\underline{m}_2 = \Lambda\bar{e}_1 \oplus \Lambda\bar{\gamma} \oplus \Lambda\bar{e}_3 \oplus \Lambda\bar{e}_4$

$\underline{m}_3 = \Lambda\bar{e}_1 \oplus \Lambda\bar{e}_2 \oplus \Lambda\bar{\delta} \oplus \Lambda\bar{e}_4$

$\underline{m}_4 = \Lambda\bar{e}_1 \oplus \Lambda\bar{e}_2 \oplus \Lambda e_3 \oplus (0).$

$\underline{m}_1 \cap \underline{m}_2 \cap \underline{m}_3 \cap \underline{m}_4 = \langle \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \rangle \overset{?}{=} \operatorname{rad}\Lambda$

Proposition 16

For any ring Λ and any $\lambda \in \Lambda$, the following are equivalent

(i) $\lambda \in \operatorname{rad} \Lambda$,

(ii) $1 - x\lambda$ is left invertible for all $x \in \Lambda$ (i.e. $\exists x' \in \Lambda$ s.th $x'(1 - x\lambda) = 1$).

(iii) $\lambda \cdot S = (0)$ for any simple Λ-module S.

Proof: (i) $\Rightarrow$ (ii): Suppose $\exists x \in \Lambda$ such that $1 - x\lambda$ is not left invertible. Then $\Lambda(1 - x\lambda)$ is a proper left ideal. Any proper left ideal is contained in a maximal ideal, hence $\Lambda(1 - x\lambda) \subseteq \underline{m}$ - max. left ideal. If $\lambda \in \underline{m}$, then $1 \in \underline{m}$ $\times$ So $\lambda \notin \underline{m}$ and in particular $\lambda \notin \operatorname{rad} \Lambda$.

(ii) $\Rightarrow$ (iii): Suppose $\exists$ a simple Λ-module S such that $\lambda S \neq (0)$, i.e. $\exists\, 0 \neq s \in S$ with $\lambda s \neq 0$. Have $(0) \neq \Lambda(\lambda s) \subseteq S$. S simple $\Rightarrow \Lambda\lambda s = S$. Hence, $\exists x \in \Lambda$ such that $x\lambda s = s \Longrightarrow$

$$(1 - x\lambda)s = 0$$

If $(1 - x\lambda)$ is left invertible, then $s = 0$ $\times$

$\Rightarrow 1 - x\lambda$ is not left invertible.

(iii) $\Rightarrow$ (i): Let $\underline{m}$ be a maximal left ideal in Λ. Then $\Lambda/\underline{m}$ is a simple left Λ-module. By assumption

$$\lambda \cdot \Lambda/\underline{m} = (0).,$$

in particular

$$\lambda(1 + \underline{m}) = \lambda + \underline{m} = \overline{0}$$

and $\lambda \in \underline{m}$ for all maximal left ideals $\underline{m}$ in Λ. Hence $\lambda \in \operatorname{rad} \Lambda$.

$\square$

DEF: Let M be a (left) Λ-module, and let

$$\operatorname{Ann}_\Lambda M = \{\lambda \in \Lambda \mid \lambda m = 0, \forall m \in M\}.$$

Note: $\operatorname{Ann}_\Lambda M$ is a two-sided ideal in Λ

Corollary 17 Given a ring Λ

$$\operatorname{rad}\Lambda = \bigcap_{\substack{S \text{ simple} \\ \text{left } \Lambda\text{-module.}}} \operatorname{Ann}_\Lambda S$$

In particular, $\operatorname{rad}\Lambda$ is a two-sided ideal in Λ.

Proof: Follows from (i) $\Leftrightarrow$ (iii) in Proposition 16. $\square$

Can we find $\operatorname{rad}\Lambda$ from this?

$S \simeq S' \rightarrow \operatorname{Ann}_\Lambda S = \operatorname{Ann}_\Lambda S'$

Theorem 18 (Nakayama Lemma)
Given a ring Λ and a finitely gen. Λ-module M. If $\underline{a}$ is an ideal in Λ with $\underline{a} \subseteq \operatorname{rad}\Lambda$, then $\underline{a}M = M$ implies that $M = (0)$.

Proof: Suppose that $M \neq (0)$. and $\underline{a}M = M$. Let $\{m_1, m_2, \ldots, m_t\}$ be a minimal set of generators for M as a Λ-module. Since $\underline{a}M = M$, we have that

$$m_1 = \sum_{i=1}^{t} \lambda_i m_i \;,\; \lambda_i \in \underline{a} \subseteq \operatorname{rad}\Lambda$$

$$\Rightarrow (1-\lambda_1)m_1 = \sum_{i=2}^{t} \lambda_i m_i$$

Since $\lambda_1 \in \underline{a} \subseteq \operatorname{rad}\Lambda \xrightarrow{\text{Prop16}} 1-\lambda_1$ has a left inverse, say u.

$$\Rightarrow m_1 = u(1-\lambda_1)m_1 = \sum_{i=2}^{t} u\lambda_i m_i$$

$\Rightarrow M$ can be generated by $\{m_2, \ldots, m_t\}$ ⨳

If $t=1$, then $M=(0)$ ⨳. If $t>1$, then we have a contradiction to the choice of generating set $\{m_1, m_2, \ldots, m_t\}$

$\Rightarrow \underline{a}M \neq M$. $\square$

__Recall__: A left ideal $\underline{a} \subseteq \Lambda$ is __nilpotent__ if $\exists n \geq 1$ such that $\underline{a}^n = (0)$.

__Lemma 19__ Λ ring.

(a) If Λ is a left (right) artinian, then $\operatorname{rad}\Lambda$ is nilpotent.

(b) If $\underline{a} \subseteq \Lambda$ is a nilpotent left ideal, then $\underline{a} \subseteq \operatorname{rad}\Lambda$.

__Proof__: (a) $\underline{r} = \operatorname{rad}\Lambda$

$\cdots \supseteq \underline{r}^i \supseteq \underline{r}^{i+1} \supseteq \cdots\cdots$ descending chain of left ideals in Λ

Λ left artinian $\Rightarrow$ $\underline{r}^m = \underline{r}^{m+1} = \cdots$ for some m

$$\underset{M}{\underset{\shortparallel}{\underline{r}^m}} = \underline{r}^{m+1} = \underline{r} \cdot \underline{r}^m = \underline{r} \cdot M$$

Λ left artinian $\Rightarrow$ Λ left noetherian

$\underline{r}^m = M \subseteq \Lambda$ left ideal $\Rightarrow$ $M = \underline{r}^m$ fin. gen. Λ-module

Nakayama Lemma $\Rightarrow$ $\underline{r}^m = (0)$ and $\operatorname{rad}\Lambda$ is nilpotent.

(b) Assume that $\underline{a}$ is a nilpotent left ideal in Λ, $\underline{a}^n = (0)$ for some $n \geq 1$. Let $a \in \underline{a}$. Then for all $x \in \Lambda$ $xa \in \underline{a}$ and $(xa)^n = 0$.

$$\Rightarrow (1 + xa + (xa)^2 + \cdots + (xa)^{n-1})(1 - xa) = 1 - \overset{0}{\overset{\shortparallel}{(xa)^n}} = 1$$

$\Rightarrow$ $1 - xa$ has a left inverse $\forall x \in \Lambda$.

Prop 16 $\Rightarrow$ $a \in \operatorname{rad}\Lambda$ $\Rightarrow$ $\underline{a} \subseteq \operatorname{rad}\Lambda$ $\square$

__Recall__: Λ semisimple

$\Leftrightarrow$ ${}_{\Lambda}\Lambda$ semisimple Λ-module

$\Leftrightarrow$ $\Lambda \simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_t}(D_t)$
$n_i \geq 1$, D_i division ring, $t < \infty$.

$\Leftrightarrow$ Λ left artinian and has no nilpotent (left) ideals.

__Theorem 20__ Λ ring

Λ semisimple $\Leftrightarrow$ Λ left artinian and $\operatorname{rad}\Lambda = (0)$.

Proof: $\Rightarrow$:

$$\begin{tikzcd} \Lambda \text{ semisimple} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \Lambda \text{ left artinian.} \arrow[d, Rightarrow] \\ & \operatorname{rad}\Lambda \text{ is nilpotent} \arrow[d, Rightarrow] \\ \Lambda \text{ has no non-zero nilpotent left ideals.} \arrow[r, Rightarrow] & \operatorname{rad}\Lambda = (0). \end{tikzcd}$$

$\Leftarrow$: Assume that Λ is left artinian with $\operatorname{rad}\Lambda = (0)$.

Lemma 19(b) $\Rightarrow$ Λ has no non-zero nilpotent left ideals.

$\Rightarrow$ Λ is semisimple. $\square$

Theorem 21 Λ left artinian, $\underline{r} = \operatorname{rad}\Lambda$.
Then
(a) $\Lambda/\underline{r}$ is a semisimple ring.
(b) A left Λ-module M is semisimple if and only if $\underline{r}M = (0)$.
(c) The are only finitely many non-isomorphic simple Λ-modules, and they all occur as direct summands of $\Lambda/\underline{r}$.
(d) Λ is left noetherian.

Proof: (a) Λ left artinian
$\Rightarrow \Lambda/\underline{r}$ left artinian
$\operatorname{rad}(\Lambda/\underline{r}) = (\operatorname{rad}\Lambda)/\underline{r} = (0)$.

Theorem 20 $\Rightarrow \Lambda/\underline{r}$ is semisimple

(b)-(d): Exercise, see book Prop 3.1, page ?. $\square$

Recall Λ left artinian $\Leftrightarrow \ell({}_\Lambda\Lambda) < \infty$.

Corollary 22 Λ ring, TFAE
(a) Λ left artinian,
(b) Every finitely generated Λ-module has finite length.
(c) $\underline{r} = \operatorname{rad}\Lambda$ is nilpotent and $\underline{r}^i/\underline{r}^{i+1}$ is finitely generated semisimple Λ-module for all $i \geq 0$.

Proof: (b) $\Rightarrow$ (a): In particular, Λ as a left Λ-module has finite length.

$\Rightarrow \Lambda$ is left artinian (and left noeth)

(a) $\Rightarrow$ (c): $\underline{r}$ is nilpotent by Lemma 19(a), say $\underline{r}^n = (0)$.

Theorem 21(d) $\Rightarrow \Lambda$ left noetherian.

$\Rightarrow \underline{r}^i$ fin. gen. $\forall i$ (as a left ideal)

$\Rightarrow \underline{r}^i/\underline{r}^{i+1}$ fin. gen. $\forall i$ (as a left Λ-module)

Theorem 21(b) $\Rightarrow \underline{r}^i/\underline{r}^{i+1}$ semisimple Λ-module $\forall i \geq 0$, since $\underline{r} \cdot \underline{r}^i/\underline{r}^{i+1} = (0)$

(c) $\to$ (b): Suppose $\underline{r}^n = (0)$ for some $n \geq 1$.

Consider: $\Lambda \supseteq \underline{r} \supseteq \underline{r}^2 \supseteq \underline{r}^3 \supseteq \cdots \supseteq \underline{r}^{n-1} \supseteq \underline{r}^n = (0)$

In particular, we have exact sequences

$$0 \to \underline{r}^n \hookrightarrow \underline{r}^{n-1} \longrightarrow \underline{r}^{n-1}/\underline{r}^n \to 0$$

$$0 \to \underline{r}^{n-1} \hookrightarrow \underline{r}^{n-2} \longrightarrow \underline{r}^{n-2}/\underline{r}^{n-1} \to 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$0 \to \underline{r}^2 \hookrightarrow \underline{r} \longrightarrow \underline{r}/\underline{r}^2 \to 0$$

$$0 \to \underline{r} \hookrightarrow \Lambda \longrightarrow \Lambda/\underline{r} \to 0$$

WTS: $\ell(\underline{r}^i/\underline{r}^{i+1}) < \infty$ for all $i \geq 0$ $(\Rightarrow \ell({}_\Lambda\Lambda) < \infty)$

$\underline{r}^i/\underline{r}^{i+1}$ fin. gen. $\Rightarrow \exists t \geq 1$ and onto map

$$\begin{tikzcd}
\Lambda^t \arrow[r, "f"] \arrow[d] & \underline{r}^i/\underline{r}^{i+1} \\
\Lambda^t/\underline{r}\Lambda^t \arrow[ur, "\bar{f}"] \arrow[r, phantom, "\simeq"] & (\Lambda/\underline{r})^t \arrow[u, "\bar{\bar{f}}"']
\end{tikzcd}$$

, since $\underline{r}(\underline{r}^i/\underline{r}^{i+1}) = (0)$

$$0 \longrightarrow U \longrightarrow (\Lambda/\underline{r})^t \longrightarrow \underline{r}^i/\underline{r}^{i+1} \to 0 \quad \text{exact}$$

finite length?

$\Lambda/\underline{r}$ semisimple $\Rightarrow \Lambda/\underline{r} \simeq \bigoplus_{i=1}^m S_i$, $m < \infty$, S_i simple

$\to \ell(\Lambda/\underline{r}) < \infty \Rightarrow \ell((\Lambda/\underline{r})^t) < \infty$, $t < \infty$

$\Rightarrow \ell(\underline{r}^i/\underline{r}^{i+1}) < \infty$.

$\Rightarrow \ell({}_\Lambda\Lambda) < \infty$.

M fin. gen. $\Rightarrow \exists\ 0 \to \ker g \to \Lambda^\ell \to M \to 0$ exact with $\ell < \infty$

Know: $\ell({}_\Lambda\Lambda^\ell) < \infty$

$\Rightarrow \ell(M) < \infty$. $\square$

Exercise: Λ fin. dim k-algebra, k field.

M fin. gen Λ-module $\iff \ell(M) < \infty$

$\iff \dim_k M < \infty$.

Theorem 23

Let Λ be left artinian and $\underline{a} \subseteq \Lambda$ a nilpotent ideal. Then

$\Lambda/\underline{a}$ is semisimple $\iff$ $\underline{a} = \operatorname{rad}\Lambda$.

Proof: $\Leftarrow$: Theorem 21 (a) $\Rightarrow \Lambda/\underline{a}$ is semisimple

$\Rightarrow$: Assume that $\Lambda/\underline{a}$ is semisimple

$\underline{a}$ nilpotent $\xRightarrow{\text{Lemma 19 (b)}}$ $\underline{a} \subseteq \operatorname{rad}\Lambda$.

Have: $\operatorname{rad}(\Lambda/\underline{a}) = (\operatorname{rad}\Lambda)/\underline{a}$, since $\underline{a} \subseteq \operatorname{rad}\Lambda$,

max. ~~left~~ ideals in $\Lambda/\underline{a}$: $\underline{m}/\underline{a}$
$\underline{m}$ max. left ideal in Λ with $\underline{a} \subseteq \underline{m}$.

$$\left.\begin{array}{l}\Lambda/\underline{a} \text{ left artinian} \\ \text{Theorem 20}\end{array}\right\} \Rightarrow \underset{(0)}{\overset{}{\operatorname{rad}(\Lambda/\underline{a})}}$$

$\Rightarrow \underline{a} = \operatorname{rad}\Lambda$. $\square$

Examples

(1) Γ

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$

$\rho = \{\gamma\alpha - \delta\beta\}$, k field

$\Lambda = k\Gamma/\langle\rho\rangle$

Let $J = \langle \alpha, \beta, \gamma, \delta \rangle \leq k\Gamma$ - ideal gen. by all arrows,

and consider $\overline{J} = J/\langle\rho\rangle \leq \Lambda$.

Have seen: $\overline{J} = \underline{m}_1 \cap \underline{m}_2 \cap \underline{m}_3 \cap \underline{m}_4$. for some max. left ideals $\underline{m}_i$.

Is $\overline{J} = \operatorname{rad}\Lambda$?

- $k\Gamma$ fin. dim k-alg. $\Rightarrow \Lambda$ fin. dim k-alg.
  $\Rightarrow \Lambda$ left artinian.
- $\overline{J}$ nilpotent:
  $\overline{J}^3 = (\underbrace{J^3}_{\text{all paths of length} \geq 3} + \langle\rho\rangle)/\langle\rho\rangle = (0)$.
- $\Lambda/\overline{J} = k\Gamma/\langle\rho\rangle \big/ J/\langle\rho\rangle \simeq k\Gamma/J \simeq \underbrace{k^4}_{\text{semisimple}}$

Theorem 23 $\Rightarrow \operatorname{rad}\Lambda = \overline{J}$.

2) In general: Given (Γ, ρ) quiver with relations ρ over a field k, let $J \subseteq k\Gamma$ be the ideal generated by the arrows. Assume that

$$J^t \subseteq \langle \rho \rangle \subseteq J^2$$

for some $t \geq 2$ (relations are admissible.)

Let $\Lambda = k\Gamma / \langle \rho \rangle$.

Exercise: $\dim_k k\Gamma / J^t < \infty$.

Have: $\Lambda \simeq (k\Gamma / J^t) \big/ (\langle \rho \rangle / J^t)$

i.e, Λ is a factor of $k\Gamma / J^t$

$\Longrightarrow \Lambda$ fin. dim k-alg.

$\Longrightarrow \Lambda$ left artinian.

- $\bar{J} = J/\langle \rho \rangle$ nilpotent:

$$\bar{J}^t = (J^t + \langle \rho \rangle)/\langle \rho \rangle = \langle \rho \rangle / \langle \rho \rangle = (0).$$

- $$\Lambda / \bar{J} = (k\Gamma/\langle \rho \rangle) \big/ (J/\langle \rho \rangle) \simeq k\Gamma / J$$
$$\simeq \underset{\text{semisimple}}{\underbrace{k^{|\Gamma_0|}}}$$

Theorem 23 $\Longrightarrow \bar{J} = \operatorname{rad} \Lambda$.

Proposition 24

Let (Γ, ρ) and k be as above.

Then
$$\operatorname{rad} k\Gamma / \langle \rho \rangle = J / \langle \rho \rangle = \bar{J}.$$

Recall: Theorem 21 (c): Λ left artinian: Only finitely many non-isom. simple Λ-modules, and they all occur as direct summands of $\Lambda / \operatorname{rad} \Lambda$.

For example (2): $\Lambda / \bar{J} \simeq k\bar{e}_1 \oplus \cdots \oplus k\bar{e}_n$, $n = |\Gamma_0|$.

$k\bar{e}_1 \rightsquigarrow k$ in vertex 1, all $f_\alpha = 0$.

Radical of a module

DEF: Λ ring, $A \subseteq B$ Λ-modules. A is *small* in B, if

$$A + X = B$$

implies that $X = B$ for every submodule X of B.

Examples

(1) $\Lambda = \mathbb{Z}$ and $B = \Lambda$. Then the only small submodule of B is (0).

If $(0) \neq A \subsetneq B$ submodule, then

$$A = \mathbb{Z}n$$

for some $n \neq 0, 1$. Choose an integer $m \neq 0, 1$ such that $\gcd(n, m) = 1$.

Then

$$B = \mathbb{Z} = \mathbb{Z}n + \mathbb{Z}m$$
$$= A + \overset{\shortparallel}{X}$$

but $X \neq B$. Hence A is not small.

(2) Γ: $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$,

$B = k\Gamma e_1$

$$B = k\Gamma e_1 \rightsquigarrow \underbrace{\begin{tikzcd} k \arrow[d, "1"] \\ k \arrow[d, "1"] \\ k \end{tikzcd} \supseteq \begin{tikzcd} 0 \arrow[d] \\ k \arrow[d, "1"] \\ k \end{tikzcd} \supseteq \begin{tikzcd} 0 \arrow[d] \\ 0 \arrow[d] \\ k \end{tikzcd} \supseteq \begin{tikzcd} 0 \arrow[d] \\ 0 \arrow[d] \\ 0 \end{tikzcd}}$$

(the third: $\nearrow A$.)

No other proper subrep.

(i) $A + (0) = A \neq B$

(ii) $A + A = A \neq B$

(iii) $A + \begin{tikzcd} 0 \arrow[d] \\ k \arrow[d, "1"] \\ k \end{tikzcd} = \begin{tikzcd} 0 \arrow[d] \\ k \arrow[d, "1"] \\ k \end{tikzcd} \neq B$

$\Rightarrow$ A is small in B.

3) Γ $$\begin{tikzcd} & 1 \arrow[dl] \arrow[d] & 4 \arrow[dl] \\ 2 & 3 & \end{tikzcd}$$, $\Lambda = k\Gamma$, $$B = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "1"] & & k \arrow[dl, "1"] \\ k & & k & \end{tikzcd} \supseteq \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \\ k & & 0 & \end{tikzcd} = A$$

$$\cup\!\shortmid \quad \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "1"] & & 0 \arrow[dl] \\ k & & k & \end{tikzcd} = A'$$

Exercise:
- A small in B
- A' not small in B.

DEF: Λ ring, B Λ-module

$$\operatorname{rad} B = \bigcap_{\substack{A \text{ maximal} \\ \text{submodule of } B}} A = \text{the radical of } B.$$

Note: $\operatorname{rad}_\Lambda \Lambda$ = the radical of Λ as a ring.

Proposition 25 Λ ring, B fin. gen. Λ-module
$A \subseteq B$ is small in $B \iff A \subseteq \operatorname{rad} B$

Proof: $\Leftarrow$: Assume that $A \subseteq \operatorname{rad} B$. Let $X \subsetneq B$. WTS: $A + X \neq B$.
Consider $\mathcal{F} = \{M \mid M \subsetneq B \text{ submodule}, X \subseteq M\}$.
$\mathcal{F} \neq \emptyset$, since $X \in \mathcal{F}$. Let $\{C_\alpha\}_{\alpha \in I}$ be a chain of submodules of B ← Let in $\mathcal{F}$.

$$U = \bigcup_{\alpha \in I} C_\alpha,$$

is a submodule of B. If $U = B$, each element in a set of generators $\{b_1, b_2, \dots, b_n\}$ for B must be in one C_{α_i}, say $b_i \in C_{\alpha_i}$. The chain condition implies that $\{b_1, b_2, \dots, b_n\} \subseteq C_\alpha$ for some $\alpha \in I$. $\Rightarrow C_\alpha = B$ ⨳

$\Rightarrow$ Each chain in $\mathcal{F}$ has a upper bound in $\mathcal{F}$.

Zorn's Lemma $\Rightarrow$ $\mathcal{F}$ has a maximal element B_1, i.e. B_1 is a maximal submodule of B.

Then $A \subseteq \operatorname{rad} B \subseteq B_1$ and $X \subseteq B_1$, so that $A + X \subseteq B_1 \subsetneq B$

Hence A is small in B.

$\Rightarrow$: Suppose that $A \nsubseteq \operatorname{rad} B$, that is, $\exists$ maximal submodule $B_1 \subseteq B$ such that $A \nsubseteq B_1$. Then

$B_1 \subsetneq A + B_1 \subseteq B$, and consequently
$A + B_1 = B$ (since B_1 is maximal)
$B_1 \subsetneq B \Rightarrow A$ is not small in B. $\square$

Theorem 26 Λ left artinian, A fin. gen. Λ-mod. Then $\operatorname{rad} A = \underline{r} \cdot A$, where $\underline{r} = \operatorname{rad} \Lambda$.

Proof: ① $\underline{r}A \subseteq \operatorname{rad} A$: WTS: $\underline{r}A$ small in A ($\xRightarrow{\text{Prop 25}}$ $\underline{r}A \subseteq \operatorname{rad} A$).

Let X be a submodule of A and suppose that $\underline{r}A + X = A$.

$$\Longrightarrow \underline{r}^2 A + \underline{r}X = \underline{r}A$$

$$\Longrightarrow \underline{r}^2 A + \underbrace{\underline{r}X + X}_{\overset{\shortparallel}{X}} = A$$

$$\underline{r}^2 A + X = A$$

Induction: $\underline{r}^n A + X = A$ for all $n \geq 1$.

Lemma A $\Rightarrow$ $\underline{r}$ nilpotent $\Rightarrow$ $X = A$

$\Rightarrow \underline{r}A$ small in A $\Rightarrow \underline{r}A \subseteq \operatorname{rad} A$

② $\operatorname{rad} A \subseteq \underline{r}A$: $A/\underline{r}A$ is a semisimple Λ-module, since $\underline{r}(A/\underline{r}A) = (0)$ (Theorem 21(b))

Have: $A/\underline{r}A = \bigoplus_{i=1}^{t} S_i$, S_i simple Λ-module

Let $A_j = \bigoplus_{\substack{i=1 \\ i \neq j}}^{t} S_i \subseteq A/\underline{r}A$, which is a maximal submodule. Furthermore

$$\bigcap_{j=1}^{t} A_j = (0), \text{ so that } \operatorname{rad}(A/\underline{r}A) = (0).$$

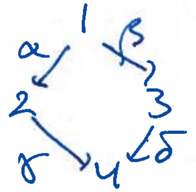
① $\Rightarrow \underline{r}A$ is contained in all maximal submodules of A.

$$\Rightarrow \operatorname{rad}(A/\underline{r}A) = (\operatorname{rad} A + \underline{r}A)/\underline{r}A = (0)$$

$$\Rightarrow \operatorname{rad} A \subseteq \underline{r}A$$

$$\Rightarrow \operatorname{rad} A = \underline{r} \cdot A. \qquad \square$$

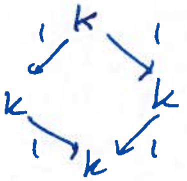
Example Γ:



, $\rho = \gamma\alpha - \delta\beta$, k field, $\Lambda = k\Gamma/\langle \rho \rangle$.

Have seen: $\underline{r} = \overline{\langle \text{arrows} \rangle} = \overline{J}$

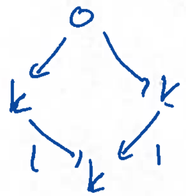
$\Lambda \bar{e}_1 \rightsquigarrow$



$\cup\!\!\mid$

$\operatorname{rad} \Lambda \bar{e}_1 = \underline{r}(\Lambda \bar{e}_1) \rightsquigarrow$

$\overset{\shortparallel}{\underline{r}\bar{e}_1}$



The radical of representation.

(Γ, ρ) quiver with relations, $J^t \subseteq \langle \rho \rangle \subseteq J^2$

k field, $\Lambda = k\Gamma / \langle \rho \rangle$, $\Gamma_0 = \{1, 2, \ldots, n\}$.
$\underset{\underline{r} = \overline{J}}{}$

(V, f) repr. of $(\Gamma, \rho) \rightsquigarrow M_{(V,f)} = V(1) \oplus V(2) \oplus \cdots \oplus V(n)$

$$\begin{tikzcd}
(V,f) \arrow[r, rightsquigarrow] & M_{(V,f)} \\
(V', f') \arrow[u, phantom, "\cup" rotate=90] & \underline{r} M_{(V,f)} \arrow[l, leftsquigarrow] \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

$\underline{r} M_{(V,f)} = \{ r_1 m_1 + \cdots + r_t m_t \mid r_i \in \underline{r}, m_i \in M \}$

(V', f') ↖ the radical of the repr. (V,f)

$\underline{r} M_{(V,f)}$ ↑ the radical of $M_{(V,f)}$

$\underline{r}$ generated by the arrows.

$\Rightarrow \underline{r} M_{(V,f)}$ is generated by elt's of the form; for $\beta : r \longrightarrow s \in \Gamma_1$.

$$\beta \cdot (v_1, v_2, \ldots, v_n) = (0, \ldots, 0, \underset{\uparrow \, s\text{-th coord.}}{f_\beta(v_r)}, 0, \ldots, 0)$$

$$\Rightarrow \bar{e}_s \underline{r} M_{(V,f)} = \sum_{\substack{\beta \in \Gamma_1 \\ e(\beta) = s}} \operatorname{Im} f_\beta$$

$$\Rightarrow V'(i) = \bar{e}_i \underline{r} M_{(V,f)} = \sum_{\substack{e(\beta) = i \\ \beta \in \Gamma_1}} \operatorname{Im} f_\beta \text{ and}$$

$$f'_\alpha = f_\alpha \big|_{V'(i)} : V'(i) \longrightarrow V'(j) \text{ for } \alpha : i \to j$$

↖ the range is by def. OK, since $\operatorname{Im} f_\alpha \subseteq V'(j)$.

Examples (1) $\Gamma : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$

$$\begin{tikzcd}[row sep=small]
 & k \arrow[d, "1"] & & 0 \arrow[d] \\
\Lambda e_1 \arrow[r, leftrightsquigarrow] & k \arrow[d, "1"] & \operatorname{rad}(\Lambda e_1) \arrow[r, leftrightsquigarrow] & k \arrow[d, "1"] \\
 & k & & k
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=small]
 & 0 \arrow[d, "0"] & & 0 \arrow[d, "0"] \\
\Lambda e_2 \arrow[r, leftrightsquigarrow] & k \arrow[d, "1"] & \operatorname{rad}(\Lambda e_2) \arrow[r, leftrightsquigarrow] & 0 \arrow[d, "0"] \\
 & k & & k
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=small]
 & 0 \arrow[d, "0"] & & 0 \arrow[d] \\
\Lambda e_3 \arrow[r, leftrightsquigarrow] & 0 \arrow[d, "0"] & \operatorname{rad}(\Lambda e_3) \arrow[r, leftrightsquigarrow] & 0 \arrow[d] \\
 & k & & 0
\end{tikzcd}$$

□

(2) $\Gamma : 1 \xrightarrow{\alpha} 2 \circlearrowright \beta$, $\rho = \{\beta^2\}$.

$\Lambda = k\Gamma/\langle\rho\rangle$

$\Lambda\bar{e}_1 \rightsquigarrow k \xrightarrow{\binom{1}{0}} k^2 \circlearrowright \begin{pmatrix}0&0\\1&0\end{pmatrix}$ $\quad 0 \longrightarrow k^2 \circlearrowright \begin{pmatrix}0&0\\1&0\end{pmatrix}$

$\Lambda\bar{e}_2 \rightsquigarrow 0 \longrightarrow k^2 \circlearrowright \begin{pmatrix}0&0\\1&0\end{pmatrix}$ $\quad 0 \longrightarrow (0)\oplus k \circlearrowright 0$

Note: In general, M and N Λ-module. Then $\operatorname{rad}(M\oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$.

DEF: Λ left artinian, $\underline{r} = \operatorname{rad}\Lambda$. A fin.gen Λ-module. Then $A/\underline{r}A$ is called the top of A.

Examples (1) $\Gamma : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, $k\Gamma = \Lambda$

$$\left.\begin{array}{l} A = \Lambda e_1 : k \xrightarrow{1} k \xrightarrow{1} k \\ \underline{r}A : 0 \longrightarrow k \xrightarrow{1} k \end{array}\right\} A/\underline{r}A : k \to 0 \to 0$$

$e_1\Lambda e_1 = ke_1$

(2) $A = \Lambda\bar{e}_1 \rightsquigarrow k \xrightarrow{\binom{1}{0}} k^2 \circlearrowright \begin{pmatrix}0&0\\1&0\end{pmatrix} \supset 0 \xrightarrow{0} k^2 \circlearrowright \begin{pmatrix}0&0\\1&0\end{pmatrix}$

$A/\underline{r}A : k \longrightarrow 0 \circlearrowright 0$ $\quad \bar{e}_1\Lambda\bar{e}_1 = k\bar{e}_1$

In general, Λ left artinian, A fin.gen

$A \to A/\underline{r}A = \underset{\underset{x_1' \neq 0}{\cup}}{S_1} \oplus \cdots \oplus \underset{\underset{x_t' \neq 0}{\cup}}{S_t}$ - semisimple each S_i simple.

Choose $\{x_1, x_2, \dots, x_t\}$ inverse images of x_i' in A. For $a \in A$, ~~then~~ $\exists\ \lambda_i \in \Lambda$ such that

$$a - \sum_{i=1}^{t} \lambda_i x_i \in \underline{r}A$$

$$\quad\quad \| \sum_{j=1}^{n} r_j a_j,\ r_j \in \underline{r},\ a_j \in A,$$

Let $A' = \Lambda\{x_1, \dots, x_t\} \subseteq A$ be the submodule gen. by $\{x_1, x_2, \dots, x_t\}$ of A.

$\underline{r}(A/A') = A/A'$

Nakayama Lemma $\Rightarrow A/A' = (0)$

$\Rightarrow A = A'$ gen. by $\{x_1, x_2, \dots, x_t\}$

(or use $\underline{r}^m = (0)$ ....)

Lemma 27 Λ left artinian, $f: A \to B$ Λ-homomorphism, A, B fin. gen. Λ-modules. Then

$f: A \to B$ is onto $\iff \bar{f}: A/\underline{r}A \to B/\underline{r}B$ onto.

Proof: Let $f: A \to B$. Then

$$f(\underline{r}A) = \underline{r}f(A) \subseteq \underline{r}B$$

$$\begin{tikzcd}
a \arrow[d, mapsto] & A \arrow[r, "f"] \arrow[d, "p_A"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "p_B"] & b \arrow[d, mapsto] \\
a+\underline{r}A & A/\underline{r}A \arrow[r, "\bar{f}"] & B/\underline{r}B & b+\underline{r}B \\
& a+\underline{r}A \arrow[r, mapsto] & f(a)+\underline{r}B &
\end{tikzcd}$$

$\Rightarrow$: Assume that $f: A \to B$ is onto

f, p_B onto $\Rightarrow p_B \cdot f = \bar{f} \cdot p_A$ onto $\Rightarrow \bar{f}$ onto

$\Leftarrow$: Assume that $\bar{f}: A/\underline{r}A \to B/\underline{r}B$ is onto.

The elements in $\operatorname{Im}\bar{f}$ are $f(a)+\underline{r}B$ for some $a \in A$

$\Rightarrow$ Given $b \in B$, then $\exists a \in A$ such that

$$b+\underline{r}B = f(a)+\underline{r}B$$

$$\Rightarrow b-f(a) \in \underline{r}B$$

$$\Rightarrow B = \operatorname{Im} f + \underline{r}B$$

Since $\underline{r}B = \operatorname{rad} B$ (Thm. 26) is small in B (Prop. 25), then $\operatorname{Im} f = B$ and f is onto. $\square$

Note: Only used that B was fin gen.!

DEF: $f: A \to B$ is an essential epimorphism if f is an epimorphism and if $g: X \to A$ is such that $fg: X \to B$ is onto, then $g: X \to A$ is onto.

Examples (1) $f: A \oplus B \longrightarrow A$, $f(a,b) = a$, A, B Λ-modules.

- f epi OK, f ess. epi?

Consider $g: X = A \longrightarrow A \oplus B$, $g(a) = (a,0)$

Then $fg(a) = f(a,0) = a \Rightarrow fg$ onto.

If $B \neq (0)$, then g is not onto $\Rightarrow f$ is not ess. epi.

(2) $\Gamma: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$

$$A = \Lambda e_1 \rightsquigarrow \underset{A}{\begin{array}{c} k \\ \downarrow 1 \\ k \\ \downarrow 1 \\ k \end{array}} \geq \underset{B}{\begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ k \end{array}}$$

Let $f: A \to A/B$ be the natural epimorphism. Let $g: X \to A$ and assume that $g: X \to A$ is not onto.

<u>WTS</u>: fg not onto

Proper submodules of A: $\begin{array}{c} 0 \\ \downarrow \\ k \\ \downarrow 1 \\ k \end{array} \geq \begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ k \end{array} \geq \begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ 0 \end{array}$

g not onto $\Rightarrow \operatorname{Im} g \subseteq \begin{array}{c} 0 \\ \downarrow \\ k \\ \downarrow 1 \\ k \end{array}$

$$\Rightarrow \operatorname{Im} fg \subseteq f\left(\begin{array}{c} 0 \\ \downarrow \\ k \\ \downarrow 1 \\ k \end{array}\right) = \begin{array}{c} 0 \\ \downarrow \\ k \\ \downarrow 1 \\ k \end{array} \Big/ \begin{array}{c} 0 \\ \downarrow 0 \\ 0 \\ \downarrow 0 \\ k \end{array} \cong \begin{array}{c} 0 \\ \downarrow \\ k \\ \downarrow \\ 0 \end{array} \subsetneq A/B = \begin{array}{c} k \\ \downarrow 1 \\ k \\ \downarrow 0 \\ 0 \end{array}$$

$\Rightarrow fg$ not onto $\Rightarrow f$ ess. epi.

(3) Λ left artinian ring, A fin. gen Λ-module

<u>Claim</u>: $p_A: A \to A/rA$ ess. epi.

<u>Proof</u>: Let $g: X \to A$, and assume that $p_A g$ is onto.

$$\begin{tikzcd} X \arrow[r, "p_X"] \arrow[d, "g"'] & X/rX \arrow[d, "\bar{g}"] \\ A \arrow[r, "p_A"] & A/rA \end{tikzcd}$$

<u>Know</u>: $g: X \to A$ onto $\Leftrightarrow \bar{g}: X/rX \to A/rA$ onto
(Lemma 27)

$p_A g = \bar{g} p_X$ & $p_A g$ onto $\Rightarrow \bar{g}$ onto

$\Rightarrow g: X \to A$ onto

$\Rightarrow p_A$ ess. epi. $\square$

<u>Exercise</u>:

$$\left.\begin{array}{l} f: A \to B \text{ ess. epi.} \\ g: B \to C \text{ — " —} \end{array}\right\} \Rightarrow \begin{array}{l} gf: A \to C \\ \text{ess. epi.} \end{array}$$

Proposition 28 Λ left artinian A, B fin. gen. Λ-modules. Let $f : A \to B$ be onto. TFAE

(a) f is an essential epimorphism

(b) $\mathrm{Ker} f \subseteq \underline{r}A$ $\quad (\underline{r} = \mathrm{rad}\,\Lambda)$.

(c) $\bar{f} : A/\underline{r}A \longrightarrow B/\underline{r}B$ is an isomorphism

Proof: (a) $\Rightarrow$ (b): Assume that f is an ess. epi. WTS: $\mathrm{Ker} f$ is small in A, i.e. $\mathrm{Ker} f \subseteq \underline{r}A$ (Prop. 25 & Thm 27).

Let $X \subseteq A$. Assume that $\mathrm{Ker} f + X = A$. Then the composition

$$X \overset{i}{\hookrightarrow} A \xrightarrow{f} B$$

is onto (proof?), such that $X \overset{i}{\hookrightarrow} A$ is onto, since f is an ess. epi.

$\Rightarrow X = A \Rightarrow \mathrm{Ker} f$ is small in A.

$\Rightarrow \mathrm{Ker} f \subseteq \underline{r}A$.

(b) $\Rightarrow$ (c): Assume that $\mathrm{Ker} f \subseteq \underline{r}A$. Have

$$A/\mathrm{Ker} f \overset{f'}{\xrightarrow{\sim}} \mathrm{Im} f = B \quad (f \text{ onto})$$
$$a + \mathrm{Ker} f \longmapsto f(a)$$

and therefore

$$\underline{r}(A/\mathrm{Ker} f) \overset{f'}{\simeq} \underline{r}B$$
$$\shortparallel$$
$$\underline{r}A/\mathrm{Ker} f$$

and

$$(A/\mathrm{Ker} f)\big/(\underline{r}A/\mathrm{Ker} f) \overset{\bar{f}'}{\simeq} B/\underline{r}B$$

$$\begin{tikzcd}
a+\underline{r}A \arrow[rr, mapsto] \arrow[ddd, mapsto, bend right] & & f(a)+\underline{r}B \\
A/\underline{r}A \arrow[rr, "\bar{f}"] \arrow[d, phantom, "\cong"] & & B/\underline{r}B \\
(A/\mathrm{Ker} f)\big/(\underline{r}A/\mathrm{Ker} f) \arrow[urr, "\bar{f}' \sim"] & \circlearrowright & f(a)+\underline{r}B \arrow[uu, equal, bend right] \\
(a+\mathrm{Ker} f)+\underline{r}A/\mathrm{Ker} f \arrow[urr, mapsto] & &
\end{tikzcd}$$

$\bar{f}$ = composition of two isomorphisms

$\Rightarrow \bar{f}$ is an isomorphism,

(c) $\Rightarrow$ (a): Assume that $\bar{f} : A/\underline{r}A \to B/\underline{r}B$ is an isomorphism. Let $g : X \to A$

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r, "f"] \arrow[d, "p_A"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "p_B"] \\
X/\underline{r}X \arrow[r, "\bar g"] & A/\underline{r}B \arrow[r, "\bar f", "\sim"'] & B/\underline{r}B
\end{tikzcd}$$

Assume that fg is onto

$\Longrightarrow \overline{fg} = \bar f \bar g : X/\underline{r}X \longrightarrow B/\underline{r}B$ onto

$\Longrightarrow \underset{\underset{\bar g}{\shortparallel}}{\bar f^{-1} \bar f \bar g} : X/\underline{r}X \longrightarrow A/\underline{r}A$ onto

Lemma 27 $\Longrightarrow g$ onto $\Longrightarrow f$ ess. epi. □

<u>Example</u>

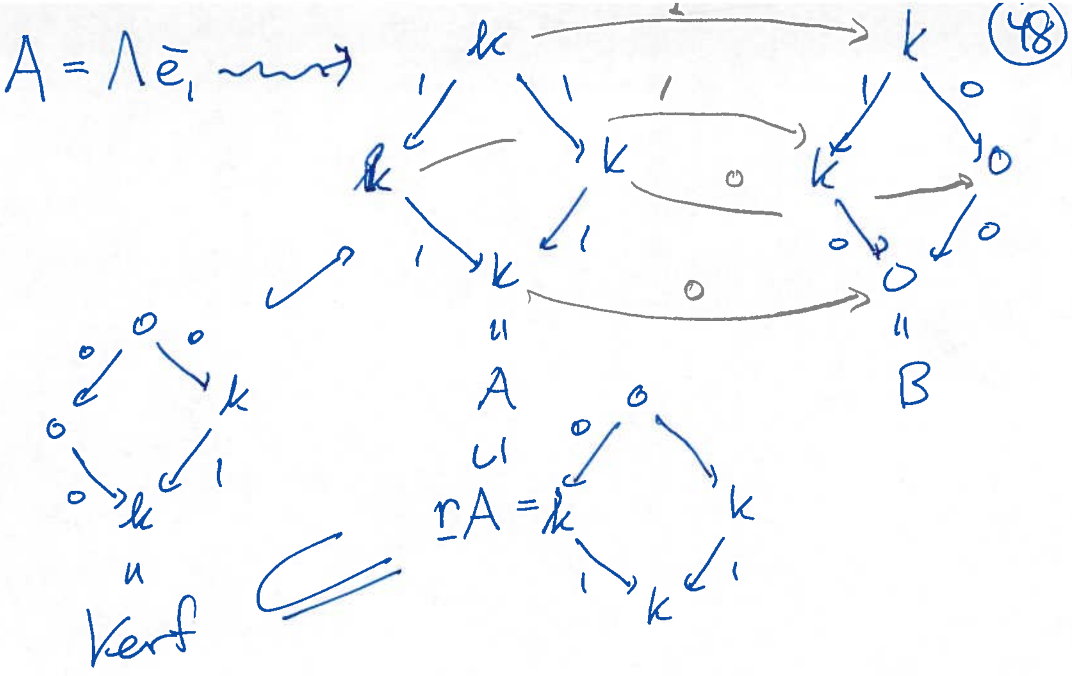
$$\Gamma : \begin{tikzcd}
& 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\
& 4 &
\end{tikzcd} \qquad \rho = \{\gamma\alpha - \delta\beta\}, \ k \text{ field}$$

$\Lambda = k\Gamma / \langle \rho \rangle$

$$B = \begin{tikzcd}
& k \arrow[dl, "1"'] \arrow[dr, "0"] & \\
k \arrow[dr, "0"'] & & 0 \arrow[dl, "0"] \\
& 0 &
\end{tikzcd}$$

Find some $f : A \to B$ which is an ess. epi.

$A = \Lambda \bar e_1 \rightsquigarrow$



$\Longrightarrow f$ ess. epi.

$$\text{or } A/\underline{r}A = \begin{tikzcd}
& k \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & & 0 \arrow[dl] \\
& 0 &
\end{tikzcd} \xrightarrow[\sim]{\bar f} \begin{tikzcd}
& k \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & & 0 \arrow[dl] \\
& 0 &
\end{tikzcd} = B/\underline{r}B$$

Projective modules

Λ ring, P Λ-module

DEF: P is projective if for every epimorphism $g: B \longrightarrow C$ of Λ-modules and every Λ-homomorphism $f: P \longrightarrow C$, there exists a homomorphism $h: P \longrightarrow B$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] \\ B \arrow[r, "g"] \arrow[ur, phantom, "\circlearrowleft" near end] & C \end{tikzcd}$$

commutes.

Example

Λ is a projective Λ-module.

$$\begin{tikzcd} & \Lambda \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & 1 \arrow[dd, mapsto] \\ B \arrow[r, "g"] & C \arrow[r] & 0 \\ b \arrow[rr, mapsto] & & f(1) \end{tikzcd}$$

$f(\lambda) = f(\lambda \cdot 1) = \lambda f(1)$

Choose $b \in B$ such that $g(b) = f(1)$, and define $h: \Lambda \longrightarrow B$ by

$$h(\lambda) = \lambda \cdot b$$

Check: h is a Λ-hom.

Then $gh(\lambda) = g(\lambda b) = \lambda g(b) = \lambda f(1)$
$= f(\lambda \cdot 1) = f(\lambda)$, $\forall \lambda \in \Lambda$

$\Longrightarrow \Lambda$ projective.

Exercise: F free Λ-module $\Rightarrow F$ is projective

$\left(F \simeq \bigoplus_{i \in I} \Lambda a_i, \ \Lambda a_i \simeq_{\Lambda} \Lambda, \forall i \in I\right)$

Proposition 29 Λ ring, P Λ-module.

P projective $\Leftrightarrow$ $\begin{cases} \exists \text{ free } \Lambda\text{-module } F \\ \text{and a } \Lambda\text{-module } Q \\ \text{such that} \\ \quad F \simeq P \oplus Q \end{cases}$

Proof: $\Rightarrow$: Assume that P is projective.

Any module M is a factor of a free module F_M: Let $F_M = \bigoplus_{m \in M} \Lambda_m$, $\Lambda_m \simeq_{\Lambda} \Lambda$

and define $\varphi: F_M \longrightarrow M$ by $\varphi((\lambda_m)_{m \in M}) = \sum_{m \in M} \lambda_m m$

<u>Check:</u> φ Λ-hom, φ onto

$$m = \varphi((\lambda_x))_{x\in M}, \text{ where } \lambda_x = \begin{cases} 0, & x \neq m \\ 1, & x = m \end{cases}$$

Let $g: F \longrightarrow P$ be onto with F free

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, equal, "1_P"] \arrow[l, phantom, "\circlearrowleft"] & \text{proj} \arrow[l] \\ F \arrow[r, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow \exists h: P \to F$ such that $gh = 1_P$

<u>Claim:</u> $F \simeq \underset{\underset{P}{\parallel}}{\operatorname{Im} g} \oplus \operatorname{Ker} h$ (Exercise)

<u>$\Leftarrow$</u>: Assume that $F \xrightarrow[\sim]{\varphi} P \oplus Q$ where F is a free Λ-module. Suppose that $g: B \longrightarrow C$ is onto, and let $f: P \longrightarrow C$.

$(p,q) \longmapsto p$

$$\begin{tikzcd} F \arrow[r, "\varphi", "\sim"'] \arrow[d, dotted, "\exists h'"'] & P \oplus Q \arrow[r, shift left, "\pi"] \arrow[d, phantom, "\circlearrowleft"] & P \arrow[l, dashed, shift left, "\nu"] \arrow[d, "f"] \\ B \arrow[rr, "g"] & & C \arrow[r] & 0 \end{tikzcd}$$

$\nu(p) = (p,0)$

$\pi\nu(p) = \pi(p,0)$
$= p$

F proj $\Rightarrow \exists h': F \to B$ such that

$$gh' = f\pi\varphi \quad \Big| \cdot \varphi^{-1}$$
$$gh'\varphi^{-1} = f\pi\varphi\varphi^{-1} = f\pi \quad \Big| \cdot \nu$$
$$g(\underbrace{h'\varphi^{-1}\nu}_{h}) = f(\pi\nu) = f1_P = f$$

$\Longrightarrow P$ is projective. $\square$

<u>Example</u>

(Γ, ρ) quiver with relations, k field

$J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1, 2, \dots, n\}$.

$\Lambda = k\Gamma / \langle\rho\rangle$

<u>Have seen:</u> ${}_\Lambda\Lambda = \Lambda\bar{e}_1 \oplus \Lambda\bar{e}_2 \oplus \cdots \oplus \Lambda\bar{e}_n$

$\Longrightarrow \Lambda\bar{e}_i$ projective Λ-modules

DEF: Let $f: P \to M$ be a Λ-hom.
Then $f: P \to M$ is a projective cover of M, if P is projective and f is an essential epimorphism.

Example Γ
$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$
$\rho = \{\gamma\alpha - \delta\beta\}$, $\Lambda = k\Gamma/\langle\rho\rangle$

$$M = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "0"] & \\ k \arrow[dr, "0"'] & & 0 \arrow[dl, "0"] \\ & 0 & \end{tikzcd}$$

$$0 \to \operatorname{Ker} f \to \Lambda\bar{e}_1 \xrightarrow{f} M \to 0$$

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & & & k \arrow[dl, "1"'] \arrow[dr, "1"] & & & & k \arrow[dl, "1"'] \arrow[dr, "0"] & \\ 0 \arrow[dr] & & k \arrow[dl, "1"] \arrow[r, hook] & k \arrow[dr, "1"'] & & k \arrow[dl, "1"] & & k \arrow[dr, "0"'] & & 0 \arrow[dl, "0"] \\ & k & & & k & & & & 0 & \end{tikzcd}$$

Have $\operatorname{Ker} f \subseteq \underline{r}(\Lambda\bar{e}_1)$
$\Rightarrow$ f is an essential epimorphism
$\Rightarrow$ f is a projective cover.

Theorem 30 Λ left artinian, A fin. gen.
(a) $\exists$ a projective cover $f: P \to A$ (P fin. gen.)
(b) Let $f_1: P_1 \to A$ and $f_2: P_2 \to A$ be two projective covers of A. Then $\exists$ an isomorphism $g: P_1 \to P_2$ such that $f_1 = f_2 g$

$$\begin{tikzcd} P_1 \arrow[dr, "f_1"] \arrow[dd, "g"', "\wr"] & \\ \circlearrowleft & A \\ P_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

Proof: (a) A fin. gen. $\Rightarrow$ $\exists$ onto $f: \Lambda^n \to A$
$n < \infty$.
projective.
Choose $f: P \to A$ an onto Λ-hom with $\ell(P)$ minimal. WTS: f proj. cover.
Consider

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r, hook] & P \arrow[r, "f"] \arrow[d, "\pi_P"] & A \arrow[r] \arrow[d, "\pi_A"] & 0 \\ & & P/\underline{r}P \arrow[r, "\bar{f}"] & A/\underline{r}A \arrow[r] & 0 \end{tikzcd}$$

Assume that $\operatorname{Ker} f \not\subseteq \underline{r}P$. Then
$\pi_P(\operatorname{Ker} f) \neq (0)$ in $P/\underline{r}P$, which is semisimple. Know that
$$P/\underline{r}P = \pi_P(\operatorname{Ker} f) \oplus X$$
for some $X \subseteq P/\underline{r}P$.

Define $e : P/\underline{r}P \longrightarrow P/\underline{r}P$ by letting

$$e(\varpi,\omega) = (0,\omega)$$

for $(\varpi,\omega) \in P/\underline{r}P = \pi_P(\operatorname{Ker} f) \oplus X$.

Then $e^2 = e$. Consider

$$\begin{tikzcd}
 & \underline{r}P = \operatorname{Ker}\pi_P \arrow[d, hook] \\
P \arrow[r, dashed, "e'"] \arrow[d, "\pi_P"'] & P \arrow[d, "\pi_P"] \\
P/\underline{r}P \arrow[r, "e"] & P/\underline{r}P \arrow[d] \\
 & 0
\end{tikzcd}$$

$\pi_P(e' - e'e') = 0$

$\Uparrow$

Then $\pi_P e'e' = ee\pi_P = e\pi_P = \pi_P e'$

Hence, if $x = e'(1_P - e') = (1_P - e')e'$, then $\operatorname{Im} x \subseteq \underline{r}P$. Since $\operatorname{Im} x^2 \subseteq \underline{r}^2P$ and so on and $\underline{r}^t = (0)$ for some $t \geq 1$, we have that $x^t = 0$. Let $a = e'$ and $b = 1_P - e'$. Then

$$\boxed{a^tb^t = b^ta^t = x^t = 0.}$$

and $1_P = (a+b)^{2t}$

$$= \underbrace{a^{2t} + r_1a^{2t-1}b + r_2a^{2t-2}b^2 + \cdots + r_ta^tb^t}_{\tilde e}, \; \underbrace{+ r_{t+1}a^{t-1}b^{t+1} + \cdots + r_{2t-1}ab^{2t-1} + b^{2t}}_{\tilde e'}$$

It follows that

$$\tilde e = \tilde e \cdot 1_P = \tilde e(\tilde e + \tilde e') = \tilde e^2 + \underset{\overset{\shortparallel}{0}}{\tilde e\tilde e'} = \tilde e^2$$

$\Rightarrow \tilde e$ is an idempotent.

Check: $\overline{\tilde e} = e^{2t} = e$ and

$$\bar f(P/\underline{r}P) = \bar f(X) = \underline{\bar f(\operatorname{Im} e)}.$$

Then we have

$$\begin{tikzcd}
P \arrow[r, "\tilde e"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[r, "f"] \arrow[d, "\pi_P"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "\pi_A"] \\
P/\underline{r}P \arrow[r, "e"] & P/\underline{r}P \arrow[r, "\bar f"] & A/\underline{r}A
\end{tikzcd}$$

$\underbrace{\bar f e \pi_P}_{\text{onto}} = \underbrace{\pi_A f \tilde e}_{\text{onto}}$ $\Rightarrow$ onto Seen: π_A ess. epi.

$\Rightarrow f\tilde e$ onto

$\Rightarrow f|_{\operatorname{Im}\tilde e} : \operatorname{Im}\tilde e \to A$ onto

We have $P = \operatorname{Im}\tilde e \oplus \underset{\operatorname{Im}(1_P - \tilde e)}{\underset{\shortparallel}{\operatorname{Ker}\tilde e}}$

Check: projective

$\overline{(1_P-\tilde{e})} = 1-e \neq 0 \Rightarrow \operatorname{Im}(1_P-\tilde{e}) \neq (0).$

$\Rightarrow \ell(\operatorname{Im}\tilde{e}) < \ell(P)$ $\times\!\!\times$.

$\Rightarrow \operatorname{Ker} f \subseteq \underline{r}P$ and f is an ess. epi.

$\Rightarrow f$ projective cover.

(b) ~~Let f_1, f_2~~ Let $f_i : P_i \longrightarrow A$ be a projective cover of A for $i=1,2$.

$$\begin{tikzcd}
 & P_1 \arrow[dl, bend right, "\exists g"'] \arrow[d, "f_1"] & 0 \\
P_2 \arrow[r, "f_2"] \arrow[d, "\exists h"'] & A \arrow[ur] \arrow[r] \arrow[d] & 0 \\
P_1 \arrow[ur, "f_1"'] & 0 &
\end{tikzcd}$$

G

G

P_1 proj. & f_2 epi

$\Rightarrow \exists g : P_1 \to P_2$ such that

$f_1 = f_2 g$

f_2 ess. epi & f_1 epi

$\Rightarrow g$ epi

Similarly, h epi.

$\Rightarrow \ell(P_2) \leq \ell(P_1) \leq \ell(P_2) \Rightarrow \ell(P_1) = \ell(P_2)$

$\Rightarrow g$ isomorphism. □

Proposition 31 Λ left artinian, $f: P \to A$ epi, A fin. gen, P projective Λ-module.

(a) $f: P \longrightarrow A$ projective cover

$\Longleftrightarrow \bar{f} : P/\underline{r}P \longrightarrow A/\underline{r}A$ isom.

(b) $\{f_i : P_i \longrightarrow A_i\}_{i=1}^{m}$, f_i onto, P_i proj, $i=1,2,\dots,m$

The map

$$f : P_1 \oplus P_2 \oplus \cdots \oplus P_m \longrightarrow A_1 \oplus A_2 \oplus \cdots \oplus A_m$$

defined by

$$(p_1, p_2, \dots, p_m) \longmapsto (f_1(p_1), f_2(p_2), \dots, f_m(p_m))$$

is a projective cover $\Longleftrightarrow$ each $f_i : P_i \longrightarrow A_i$ is a projective cover.

Proof: (a) Proposition 28.

(b) Exercise: Use that

$$\underline{r}(P_1 \oplus \cdots \oplus P_m) = \underline{r}P_1 \oplus \cdots \oplus \underline{r}P_m$$

and (a). ($\underline{r} = \operatorname{rad}\Lambda$). □

Proposition 32 Λ left artinian, fin. gen. modules.

(a) P proj. module. Then $P \to P/\underline{r}P$ is a proj. cover.

(b) P, Q proj. modules

$$P \simeq Q \iff P/\underline{r}P \simeq Q/\underline{r}Q$$

(c) P proj. module

$$P \text{ indecomposable} \iff P/\underline{r}P \text{ simple}$$

(d) Assume that $P = \bigoplus_{i=1}^{n} P_i \simeq \bigoplus_{j=1}^{m} Q_j$, with P projective, and P_i and Q_j are indec.

$\Rightarrow n = m$, and $\exists$ permutation π of $\{1, 2, \ldots, n\}$ such that

$$P_i \simeq Q_{\pi(i)} \quad \text{for } i = 1, 2, \ldots, n.$$

__Proof__: (a) __Have seen__: $A \to A/\underline{r}A$ is an ess. epi. $\forall$ fin. gen. A.

P projective $\Rightarrow P \to P/\underline{r}P$ proj. cover.

(b) __$\Rightarrow$__: Clear

__$\Leftarrow$__: Assume that $P/\underline{r}P \simeq Q/\underline{r}Q$.

(a) $\Rightarrow P \to P/\underline{r}P$, $Q \to Q/\underline{r}Q$ proj. covers

Assumption & proj. covers unique up to isom. (Thm 30(b)) $\Rightarrow P \simeq Q$.

(c) __$\Leftarrow$__: Assume that $P = P_1 \oplus P_2$ with $P_i \neq (0)$

Since $\underline{r}P = \underline{r}P_1 \oplus \underline{r}P_2$, we have that

$$P/\underline{r}P = P_1 \oplus P_2 / \underline{r}P_1 \oplus \underline{r}P_2 \simeq \underset{\substack{\neq \\ (0)}}{P_1/\underline{r}P_1} \oplus \underset{\substack{\neq \\ (0)}}{P_2/\underline{r}P_2}$$

Nakayama Lemma

$\Rightarrow P/\underline{r}P$ not simple

__$\Rightarrow$__: Assume that $P/\underline{r}P$ is not simple

$P/\underline{r}P$ is a semisimple Λ-module

$\Rightarrow P/\underline{r}P = U_1 \oplus U_2$, where $U_i \neq (0)$, $i = 1, 2$

Let P_i be a projective cover of U_i for $i = 1, 2$.

(a) $P \to P/\underline{r}P$ proj. cover.

__Know__: $P_1 \oplus P_2 \longrightarrow U_1 \oplus U_2 = P/\underline{r}P$ proj. cover.

Uniqueness (Thm 30(b))

$\Rightarrow P \simeq P_1 \oplus P_2$, with $P_i \neq (0)$ $i = 1, 2$.

$\Rightarrow P$ decomposes.

(d) <u>Exercise:</u>

Note: $P_i/\underline{r}P_i \hookrightarrow P_1/\underline{r}P_1 \oplus \cdots \oplus P_m/\underline{r}P_n$

$$\wr\wr$$

$$Q_1/\underline{r}Q_1 \oplus \cdots \oplus Q_m/\underline{r}Q_m \twoheadrightarrow Q_j/\underline{r}Q_j$$

$\Rightarrow \exists j$ such that $Q_j/\underline{r}Q_j \simeq P_i/\underline{r}P_i$

$$\Downarrow$$

$$P_i \simeq Q_j \qquad \square$$

Recall: Λ left artinian

$\Rightarrow \Lambda/\underline{r} = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, S_i simple

Furthermore, if T is a simple Λ-module, then $\exists j$ such that $T \simeq S_j$, i.e. $\{S_i\}_{i=1}^{n}$ has representatives from all isomorphism classes of simple Λ-mod's.

Let $P_i \xrightarrow{f_i} S_i$ be a proj. cover of S_i

Prop. 31 (a) $\Rightarrow$ $P_i/\underline{r}P_i \simeq S_i$, since $\underline{r}S_i = (0)$.

Prop 32 (c) $\Rightarrow$ P_i indec.

Let P be an indec. proj. Λ-module. Then $P \longrightarrow P/\underline{r}P$ is a proj. cover, and $P/\underline{r}P$ is simple. Hence $\exists j$ such that $P/\underline{r}P \simeq S_j$.

$$P \longrightarrow P/\underline{r}P \simeq S_j \longleftarrow P_j \text{ proj. covers.}$$

Uniqueness $\Longrightarrow P \simeq P_j$

<u>Corollary 33.</u> Λ left artinian.
The only indecomposable proj. Λ-modules are $P_1, P_2, \ldots, P_n$ up to isomorphism.

<u>Example</u> Γ
$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$
$\rho = \{\gamma\alpha - \delta\beta\}$, k field, $\Lambda = k\Gamma/\langle\rho\rangle$

<u>Have seen:</u> $\Lambda/\underline{r} = \underset{S_1}{\overset{\shortparallel}{k\bar{e}_1}} \oplus \underset{S_2}{\overset{\shortparallel}{k\bar{e}_2}} \oplus \underset{S_3}{\overset{\shortparallel}{k\bar{e}_3}} \oplus \underset{S_4}{\overset{\shortparallel}{k\bar{e}_4}}$

Projective covers,