Lemma 34 Λ left artinian, M fin. gen. Let $P \xrightarrow{f'} M/\underline{r}M$ be a proj. cover. Then a homomorphism $f: P \to M$ such that

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[d, "f'"] \\ M \arrow[r, "\pi_M"] & M/\underline{r}M \end{tikzcd}$$

commutes, is a projective cover.

Proof: $f' = \pi_M f$ epi
$M \xrightarrow{\pi_M} M/\underline{r}M$ ess. epi $\Bigg\} \Rightarrow f$ epi.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, equal] & M \arrow[d, "\pi_M"] \\ P \arrow[r, "f'"] \arrow[d, "\pi_P"] & M/\underline{r}M \arrow[d, equal] \\ P/\underline{r}P \arrow[r, "\overline{f}"] & M/\underline{r}M \end{tikzcd}$$

Know: $\overline{f'}: P/\underline{r}P \xrightarrow{\sim} M/\underline{r}M$ isom.

Have: $\pi_M f = f' = \overline{f'}\,\pi_P$
$\qquad\;\; \| $
$\qquad \overline{f}\,\pi_P$

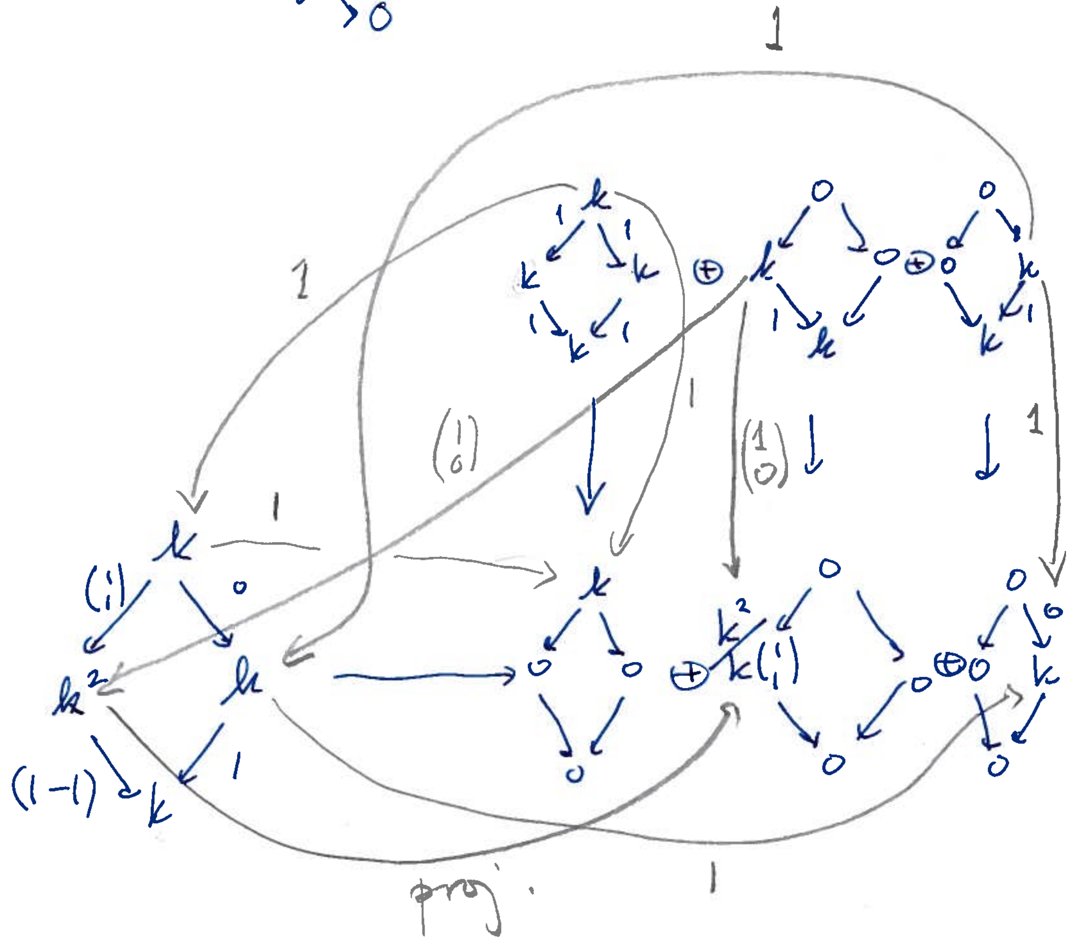
π_P epi $\Rightarrow \overline{f} = \overline{f'} \leftarrow$ isom.
$\Rightarrow \overline{f}$ isom.
$\Rightarrow f: P \to M$ proj. cover. $\square$

Example

M:
$$\begin{tikzcd} & k \arrow[dl, "\binom{1}{1}"'] \arrow[dr, "0"] & \\ k^2 \arrow[dr, "(1\,-1)"'] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd}$$

$\underline{r}M$:
$$\begin{tikzcd} & 0 \arrow[dl, "0"'] \arrow[dr, "0"] & \\ k\binom{1}{1} \arrow[dr, "0"'] & & 0 \arrow[dl, "0"] \\ & k & \end{tikzcd}$$

$M/\underline{r}M$:
$$\begin{tikzcd} & k \arrow[dl, "0"'] \arrow[dr, "0"] & \\ k^2/k\binom{1}{1} \arrow[dr, "0"'] & & k \arrow[dl, "0"] \\ & 0 & \end{tikzcd} \simeq S_1 \oplus S_2 \oplus S_3$$



DEF: Λ is a <u>local ring</u> if the non-invertible elements in Λ is an ideal.

<u>Examples</u>

(1) $\mathbb{Z}$ is not a local ring

Invertible elt's.: $\{-1,1\}$

$3+(-2)=1 \Rightarrow$ Non-invertible elt's is not an ideal.

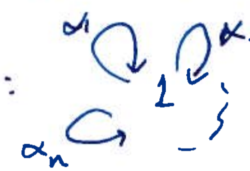
(2) k field $\Rightarrow$ k local.

(3) $\Lambda_n = k[x]/\langle x^n\rangle$, k field.

Non-invert. elt's: $\langle x\rangle/\langle x^n\rangle$

Invertible elt's: $\left\{a+f(x)x+\langle x^n\rangle \,\middle|\, \begin{array}{l} a\in k\setminus\{0\} \\ f(x)\in k[x] \end{array}\right\}$

$\Rightarrow \Lambda_n$ is a local rings.

(4) Γ:  , ρ relations such that $J^t \subseteq \langle\rho\rangle \subseteq J^2$

$\Lambda = k\Gamma/\langle\rho\rangle$ - fin. dim k-alg, k field.

Non-invert. elt's: $J/\langle\rho\rangle = \underline{r}$ - ideal

Invertible elt's: $\{ae_1 + r \mid a\in k\setminus\{0\}, r\in\underline{r}\}$

$\Rightarrow \Lambda$ is a local ring.

<u>Proposition 35</u>

Λ local ring $\Rightarrow$ 0 and 1 the only idempotents.

<u>Proof:</u> Let e be an idempotent in Λ.

Suppose that e is invertible, i.e. $\exists f\in\Lambda$ such that $ef=fe=1$.

$\Rightarrow e = e\cdot 1 = e(ef) = e^2 f = ef = 1$

Suppose that $e\neq 0,1$, i.e. e and $1-e$ are not invertible. Then

$$1 = e + (1-e)$$

non-invertible $\lightning$.

Λ local $\Rightarrow$ 1 non-invertible $\lightning$.

$\Rightarrow 0, 1$ the only idempotents in Λ $\square$

<u>Note:</u> (1) $\Lambda = k\Gamma/\langle\rho\rangle$, k field, $J^t\subseteq\langle\rho\rangle\subseteq J^2$

$|\Gamma_0|\geq 2$.

$\Rightarrow \Lambda$ not local. (Why?)

(2) Λ ring, $(0)\neq M$ Λ-module

<u>Have seen:</u> $\mathrm{End}_\Lambda(M)$ contains idempotent $\neq 0,1$

$\Leftrightarrow M$ decomposes.

Corollary 36

M Λ-module, $\mathrm{End}_\Lambda(M)$ local $\Rightarrow$ M indecomposable.

Proposition 37 Λ left artinian, P fin. gen. proj. Λ-module. TFAE

(a) P indecomposable

(b) $\mathrm{End}_\Lambda(P)$ local

(c) $\underline{r}P$ is the only maximal submodule of P.

(d) $P/\underline{r}P$ is simple.

Proof: (a) $\Leftrightarrow$ (d): Proposition 32 (c).

(d) $\Rightarrow$ (c): $P/\underline{r}P$ simple $\Rightarrow$ $\underline{r}P$ maximal submod. of P

Have: $\underline{r}P = \mathrm{rad}\, P = \bigcap_{M \text{ max. submodule of } P} M$

$\Rightarrow \underline{r}P \subseteq M$, $\forall M \subseteq P$ max. submodule.

$\Rightarrow \underline{r}P$ the only max. submodule of P.

(c) $\Rightarrow$ (b): Let $f: P \longrightarrow P$. Then

f invertible $\overset{\text{def}}{\Longleftrightarrow}$ f is an isomorphism

$\overset{(\ell(P)<\infty)}{\Longleftrightarrow}$ f is an epimorphism

$\overset{\underline{r}P \text{ max.}}{\Longleftrightarrow} \mathrm{Im} f \not\subseteq \underline{r}P$

Hence,

$\{\text{non-invert. elt's in } \mathrm{End}_\Lambda(P)\}$

$= \{f \in \mathrm{End}_\Lambda(P) \mid \mathrm{Im} f \subseteq \underline{r}P\} = I$

I ideal: $f_1, f_2 \in I$, $p \in P$

$(f_1 - f_2)(p) = \underbrace{f_1(p)}_{\in\, \underline{r}P} - \underbrace{f_2(p)}_{\in\, \underline{r}P} \in \underline{r}P \Rightarrow f_1 - f_2 \in I$

$f_1, f_2 \in \mathrm{End}_\Lambda(P)$:

$\mathrm{Im} f_1 f_2 = f_1(\mathrm{Im} f_2) \subseteq \underline{r}P$, if $f_1 \in I$

$\cap\!\!\mid$ if $f_2 \in I$

$f_1(\underline{r}P)$

$\shortparallel$

$\underline{r}\, f_1(P) \subseteq \underline{r}P$

$\Rightarrow f f_1, f_1 f \in I$, when $f_1 \in I$ and $f \in \mathrm{End}_\Lambda(P)$.

$\Rightarrow I$ is an ideal $\Rightarrow \mathrm{End}_\Lambda(P)$ local ring.

(b) $\Rightarrow$ (a): Corollary 36. $\square$

Examples

(1) $\mathbb{Z}$ projective $\mathbb{Z}$-module, $\mathbb{Z}$ not artinian

$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) \simeq \mathbb{Z} \leftarrow$ not local, but $\mathbb{Z}$ is indec.

(2) $\Lambda = k\Gamma/\langle\rho\rangle$, k field, $J^t \subseteq \langle\rho\rangle \subseteq J^2$

Λ left artinian.

Have: ~~A~~ $\Lambda\bar{e}_i$ indec proj. Λ-modules

Prop. 37 $\Rightarrow \mathrm{End}_\Lambda(\Lambda\bar{e}_i)^{op} \simeq \bar{e}_i\Lambda\bar{e}_i$ local ring.

Corollary 38 Λ left artinian. TFAE.

(a) Λ local

(b) $\mathfrak{r} = \mathrm{rad}\,\Lambda$ is a maximal left ideal

(c) $\Lambda/\mathfrak{r}$ simple left Λ-module

Proof: Follows from Prop. 37, noting that $\Lambda \simeq \mathrm{End}_\Lambda(\Lambda)^{op}$

Proposition 39 Λ left artinian

(a) $1 = e_1 + e_2 + \cdots + e_n$

- a sum of primitive orthogonal idemp

orthogonal: $e_i e_j = 0$, $\forall i \neq j$

primitive: $\begin{cases} e_i \neq 0 \text{ is not a sum of} \\ \text{two non-zero orthogonal} \\ \text{idempotents} \end{cases}$

(b) Let $e_1, e_2, \ldots, e_m$ be idempotents in Λ, and let $e = e_1 + e_2 + \cdots + e_m$.

If $e_1, e_2, \ldots, e_m$ are orthogonal, then

$$\Lambda e = \Lambda e_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_m.$$

(c) Let $e \neq 0$ be an idempotent. Then

Λe indecomposable $\Longleftrightarrow e$ is primitive.

Proof: (a) $\Lambda/\mathfrak{r}$ semisimple

$\Rightarrow \Lambda/\mathfrak{r} = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, S_i simple Λ-module

Let $P(S_i) \longrightarrow S_i$ be the proj. cover of S_i. Then $P(S_i)$ is indec by Prop. 32(c).

Then by Prop. 31 (b): $\bigoplus_{i=1}^{n} P(S_i) \longrightarrow \Lambda/\mathfrak{r}$

is a proj. cover. Also, $\Lambda \longrightarrow \Lambda/\mathfrak{r}$ is a proj. ~~cov~~ cover.

Uniqueness $\Rightarrow$ $\Lambda \simeq \bigoplus_{i=1}^{n} P(S_i)$

$\Rightarrow \Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$, P_i indec.

Then $1 = e_1 + e_2 + \cdots + e_n$ with $e_i \in P_i$

$$\Rightarrow \underset{P_i}{\underset{\cap}{e_i}} = e_i \cdot 1 = \underset{P_1}{\underset{\cap}{e_i e_1}} + \underset{P_2}{\underset{\cap}{e_i e_2}} + \cdots + \underset{P_i}{\underset{\cap}{e_i e_i}} + \cdots + \underset{P_n}{\underset{\cap}{e_i e_n}}$$

Direct sum $\Rightarrow$ $e_i e_j = \begin{cases} 0, & \text{if } i \neq j \\ e_i, & \text{if } i = j \end{cases}$

$\Rightarrow \{e_i\}$ orthogonal idempotents.

<u>Claim</u>: $P_i = \Lambda e_i$

$e_i \in P_i \Rightarrow \Lambda e_i \subseteq P_i$

$$x \in P_i = x = x \cdot 1 = \underset{P_1}{\underset{\cap}{xe_1}} + \underset{P_2}{\underset{\cap}{xe_2}} + \cdots + \underset{P_i}{\underset{\cap}{xe_i}} + \cdots + \underset{P_n}{\underset{\cap}{xe_n}}$$

Direct sum $\Rightarrow x = xe_i \in \Lambda e_i \Rightarrow P_i = \Lambda e_i$

<u>e_i primitive</u>: Let $e_i = f_1 + f_2$ with $f_i \neq 0$ for $i = 1, 2$ and f_1 and f_2 are orthogonal idempotents. Using (b):

(60)

$$\Lambda e_i = \underset{\neq 0}{\Lambda f_1} \oplus \underset{\neq 0}{\Lambda f_2}$$

$\Rightarrow P_i$ not indec. $\lightning$.

$\Rightarrow e_i$ primitive for $i = 1, 2, \ldots, n$. $\square$

<u>(b) and (c)</u>: Exercises.

<u>Proposition 40</u> Λ left artinian, P fin. gen. proj. Λ-module. Then

$$P \simeq \bigoplus_{i=1}^{n} P_i$$

with P_i indec. and decomposition is unique up to isomorphism and ordering.

<u>Proof</u>: $P/\mathfrak{r}P = \bigoplus_{i=1}^{n} S_i$, S_i simple Λ-modules

$P \longrightarrow P/\mathfrak{r}P$ proj. cover & $\bigoplus_{i=1}^{n} P(S_i) \longrightarrow \bigoplus_{i=1}^{n} S_i = P/\mathfrak{r}P$ proj. cover

Uniqueness $\Rightarrow P \simeq \bigoplus_{i=1}^{n} \underbrace{P(S_i)}_{\text{indec.}}$ $\square$

Krull-Remak-Schmidt theorem

Lemma 41 (Fitting lemma)
Λ ring, M Λ-module with $\ell(M)<\infty$, $\varphi \in End_\Lambda(M)$.
Then $\exists n \geq 1$ such that

$$M = Im\varphi^n \oplus Ker\varphi^n$$

Proof: $\ell(M)<\infty \Rightarrow M$ artinian and noetherian.

$$\Rightarrow \left\{\begin{array}{l} Im\varphi \supseteq Im\varphi^2 \supseteq \cdots \\ Ker\varphi \subseteq Ker\varphi^2 \subseteq \cdots \end{array}\right\} \text{become stationary}$$

$\Rightarrow \exists n$ such that

$$Im\varphi^n = Im\varphi^{n+1} = \cdots$$
$$Ker\varphi^n = Ker\varphi^{n+1} = \cdots$$

$$\left.\begin{array}{l} \Rightarrow \ell(Im\varphi^n) = \ell(Im\varphi^{2n}) \\ \text{Have } \varphi^n : Im\varphi^n \to Im\varphi^{2n} \text{ surjective} \end{array}\right\} \Rightarrow$$

$$\varphi^n : Im\varphi^n \underset{\psi}{\overset{\longrightarrow}{\longleftarrow}} Im\varphi^{2n} \text{ isom.}$$

Let $\psi : Im\varphi^{2n} \to Im\varphi^n$ be an inverse of φ^n.

(1) Have: $Im\varphi^n, Ker\varphi^n \subseteq M$

(2) $\underline{M = Im\varphi^n + Ker\varphi^n}$: Let $m\in M$. Then

$$m = \underbrace{\psi\varphi^n(m)}_{\in Im\varphi^n} + \underbrace{m - \psi\varphi^n(m)}_{\in Ker\varphi^n}, \text{ since}$$

$$\varphi^n(m-\psi\varphi^n(m)) = \varphi^n(m) - \underbrace{\varphi^n\psi}_{1_{Im\varphi^{2n}}}\varphi^n(m) = 0.$$

(3) $\underline{Im\varphi^n \cap Ker\varphi^n = (0)}$:
Let $m \in Im\varphi^n \cap Ker\varphi^n$. Then
$m = \varphi^n(m')$ for some $m' \in M$.
$m \in Ker\varphi^n \Rightarrow 0 = \varphi^n(m) = \varphi^{2n}(m')$
$\Rightarrow m' \in Ker\varphi^{2n} = Ker\varphi^n$
$\Rightarrow m = \varphi^n(m') = 0.$ $\square$

Theorem 42 Λ left artinian, M fin.gen. Λ-mod.
Then

$$M \text{ indec} \Longleftrightarrow End_\Lambda(M) \text{ local}$$

Proof: $\Leftarrow$: Have seen (True in general)
$\Rightarrow$: Assume that M is indec. Let

$\varphi \in \mathrm{End}_\Lambda(M)$ be non-invertible. Then $\ell(\mathrm{Im}\,\varphi) < \ell(M)$

$\Rightarrow \forall \psi \in \mathrm{End}_\Lambda(M)$ the comp. $\psi\varphi$ is not invertible ($\ell(\mathrm{Im}\,\psi\varphi) \leq \ell(\mathrm{Im}\,\varphi)$).

Fitting lemma $\Rightarrow M \cong \mathrm{Im}(\psi\varphi)^n \oplus \mathrm{Ker}(\psi\varphi)^n$

M indec. $\Rightarrow \mathrm{Im}(\psi\varphi)^n = (0)$ and $\mathrm{Ker}(\psi\varphi)^n = M$

or $\mathrm{Im}(\psi\varphi)^n = M$ and $\mathrm{Ker}(\psi\varphi)^n = (0)$.

Have $\mathrm{Im}(\psi\varphi)^n \subsetneq M$, so $\mathrm{Im}(\psi\varphi)^n = (0)$ and $\psi\varphi$ is nilpotent.

$\Rightarrow 1_M - \psi\varphi$ invertible in $\mathrm{End}_\Lambda(M)$ for all $\psi \in \mathrm{End}_\Lambda(M)$.

$\Rightarrow \varphi \in \mathrm{rad}\,\mathrm{End}_\Lambda(M) \subseteq \{$non-invertible elt's in $\mathrm{End}_\Lambda(M)\}$

$\Rightarrow \mathrm{End}_\Lambda(M)$ is local $\square$

<u>Theorem</u> 43 (Krull-Remak-Schmidt theorem) Λ left artinian, M fin. gen Λ-module.

(a) M can be written as a finite direct sum of indecomposable modules, i.e. $M = \bigoplus_{i=1}^n M_i$ with M_i indec.

(b) The decomposition of M into indec. modules is unique up to isomorphism and ordering.

<u>Proof</u>: (a) Induction on $\ell(M)$:

If $\ell(M) = 1$, then M is simple and clearly indec. Claim in (a) trivially true.

Assume (a) true for all Λ-modules X with $\ell(X) < n$. Suppose $\ell(M) = n$.

If M is indec, then we are done.

If M decomposes, say $M = M_1 \oplus M_2$ ($M_i \neq 0$) then $\ell(M_i) < \ell(M)$ for $i = 1, 2$. Then we are done by induction.

(b) Assume that

$$M = \bigoplus_{i=1}^n M_i = \bigoplus_{j=1}^m N_j$$

with M_i and N_j indec.

① If $\ell(M) = 1$, the claim is true as M is simple and therefore indec.

② Assume true for all modules X with $\ell(X) < n$. Let $\ell(M) = n$.

Denote by φ_{sr} and ψ_{rs} the compositions

$$M_r \hookrightarrow \bigoplus_{i=1}^{n} M_i = \bigoplus_{j=1}^{m} N_j \twoheadrightarrow N_s$$

and

$$N_s \hookrightarrow \bigoplus_{j=1}^{m} N_j = \bigoplus_{i=1}^{n} M_i \twoheadrightarrow M_r$$

respectively. Then

$$\sum_{s=1}^{m} \psi_{is}\varphi_{si} = 1_{M_i}$$

Since $End_R(M_i)$ is local, $\exists j$ such that $\psi_{ij}\varphi_{ji}$ is an isomorphism. If $\varphi_{ji}\psi_{ij} : N_j \to N_j$ is in $rad\,End_R(N_j)$, then it follows from Fitting Lemma that $\varphi_{ji}\psi_{ij}$ is nilpotent, say $(\varphi_{ji}\psi_{ij})^t = 0$

$\Longrightarrow (\psi_{ij}\varphi_{ji})^{t+1} = 0$ ⨳.

$\Longrightarrow \psi_{ij}\varphi_{ji}$ isomorphism

$\Longrightarrow \varphi_{ji}$ and ψ_{ij} are isomorphisms

We have

$$\begin{array}{ccc} M = \bigoplus_{r=1}^{n} M_r & \xrightarrow{1_M = (\varphi_{ji})} & \bigoplus_{s=1}^{m} N_s = M \\ \| & & \\ M_i \oplus \hat{M}_i & \xrightarrow{\begin{pmatrix} \varphi_{ji} & a \\ b & c \end{pmatrix}} & N_j \oplus \hat{N}_j \end{array}$$

$a : \hat{M}_i \to N_j$

$b : M_i \to \hat{N}_j$

$c : \hat{M}_i \to \hat{N}_j$

Check: Is 1-1 → $\begin{pmatrix} 1_{M_i} & -\varphi_{ji}^{-1}a \\ 0 & 1_{\hat{M}_i} \end{pmatrix}$ $\Big\uparrow \wr$ $\qquad \begin{pmatrix} 1_{N_j} & 0 \\ -b\varphi_{ji}^{-1} & 1_{\hat{N}_j} \end{pmatrix}$ $\Big\downarrow \wr$

= isom.

$$M_i \oplus \hat{M}_i \xrightarrow{A} N_j \oplus \hat{N}_j$$

$$A = \begin{pmatrix} 1 & 0 \\ -b\varphi_{ji}^{-1} & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \varphi_{ji} & a \\ b & c \end{pmatrix}\begin{pmatrix} 1 & -\varphi_{ji}^{-1}a \\ 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 \\ -b\varphi_{ji}^{-1} & 1 \end{pmatrix}\begin{pmatrix} \varphi_{ji} & 0 \\ b & -b\varphi_{ji}^{-1}a + c \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_{ji} & 0 \\ 0 & \underbrace{-b\varphi_{ji}^{-1}a + c}_{= \tilde{c}} \end{pmatrix} \text{ - isomorphism}$$

$\Downarrow$

φ_{ji} and $\tilde{c}$ isom.

$\Longrightarrow \tilde{c} : \hat{M}_i \to \hat{N}_j$ isomorphism, with $\ell(\hat{M}_i) < \ell(M)$.

By induction the claim is true for $\hat{M}_i$.

Since $M_i \simeq N_j$, the claim follows $\square$

Artin algebras

Recall: ① Λ fin. dim k-alg, k-field.

M fin. gen. Λ-module $\Rightarrow \dim_k M = n < \infty$

$\Rightarrow \mathrm{End}_\Lambda(M)$ fin. dim k-alg.

$\mathrm{End}_\Lambda(M) \subseteq \mathrm{End}_k(M) = M_n(k)$

② Λ left artinian, but not right artinian. $M = {}_\Lambda\Lambda$. Then $\mathrm{End}_\Lambda(M) \simeq \Lambda^{op}$ not left artinian.

R - commutative ring.

DEF: (a) An R-algebra Λ is a ring Λ and a ring homomorphism $\varphi: R \to \Lambda$ with $\mathrm{Im}\,\varphi \subseteq Z(\Lambda) = \{\lambda \in \Lambda \mid \lambda r = r\lambda, \forall r \in \Lambda\}$

(b) Λ_1, Λ_2 R-algebras given by $\varphi_1: R \to \Lambda_1$ and $\varphi_2: R \to \Lambda_2$.
Then $\psi: \Lambda_1 \to \Lambda_2$ is a homomorphism of R-algebras, if ψ is a ring hom. and

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \Lambda_1 \arrow[d, "\psi"] \\ & \Lambda_2 \end{tikzcd}$$

commutes.

(c) Λ_1 is an ~~R-alg~~ R-subalgebra of Λ_2 if Λ_1 is a subring of Λ_2 and the inclusion $\Lambda_1 \hookrightarrow \Lambda_2$ is a homomorphism of R-algebras.

Note: Λ R-algebra $\Rightarrow \Lambda$ is an R-module.

DEF: Λ is an artin R-algebra, if Λ is an R-algebra with R a commutative artinian ring and Λ is a fin. gen. R-module.

Examples

(1) Any finite dimensional algebra over a field k is an artin algebra.

(2) $k[x]$ is a k-algebra (k field) but not an artin k-algebra ($\dim_k k[x] = \infty$)

(3) R comm. artinian ring $\Rightarrow R$ artin R-alg. ($\varphi = 1_R: R \to R$, $Z(R) = R$, R gen. by 1 over R)

Note: (1) Λ R-alg. $\Rightarrow \Lambda^{op}$ R-alg.

(2) $\left.\begin{array}{l} A\ \Lambda\text{-module},\ \Lambda\ R\text{-alg} \\ \varphi: R \to \Lambda \end{array}\right\} \Rightarrow A$ is an R-module via $r \cdot a \overset{def}{=} \varphi(r)a$, $\forall r \in R$, $\forall a \in A$

(3) A, B Λ-modules, Λ R-algebra ($\varphi : R \to \Lambda$.)

$\Rightarrow \mathrm{Hom}_\Lambda(A,B)$ is an R-module via

$f \in \mathrm{Hom}_\Lambda(A,B)$, $r \in R$ $\quad (r \cdot f)(a) \overset{\text{def}}{=} \varphi(r) f(a)$.

Check this, exercise.

<u>Proposition 44</u>

Λ artin R-algebra, fin. gen left Λ-modules.

(a) A, B Λ-modules $\Rightarrow \mathrm{Hom}_\Lambda(A,B)$ fin. gen. R-module

(b) A Λ-module $\Rightarrow \mathrm{End}_\Lambda(A)$ is an artin R-algebra (which is an R-subalgebra of $\mathrm{End}_R(A)$)

(c) Λ is a left artinian ring.

<u>Proof</u>: (a) <u>Have</u>: $\cdot$ $\mathrm{Hom}_\Lambda(A,B) \subseteq \mathrm{Hom}_R(A,B)$ R-submodule

$\cdot$ R artinian $\Rightarrow$ R noetherian.

Enough to show that $\mathrm{Hom}_R(A,B)$ is a fin. gen. R-module.

A fin. gen Λ-module $\Rightarrow \exists\, \Lambda^n \to A$ onto Λ-hom, $n \geq 1$

Λ fin. gen R-module $\Rightarrow \exists\, R^m \to \Lambda$ onto R-hom, $m \geq 1$

$\Rightarrow \exists\, R^{mn} \xrightarrow{\pi} A$ onto R-hom.

$\Rightarrow \mathrm{Hom}_R(A,B) \xrightarrow{\ \nu\ } \mathrm{Hom}_R(R^{mn}, B) \simeq B^{mn}$

<u>Exercise</u>: (i) ν 1-1

(ii) $\mathrm{Hom}_R(R,B) \simeq B$ as R-modules ($f \mapsto f(1)$)

(iii) $\mathrm{Hom}_R(X \oplus Y, B) \simeq \mathrm{Hom}_R(X,B) \oplus \mathrm{Hom}_R(Y,B)$ as R-mods

Induction: $\mathrm{Hom}_R(R^{mn}, B) \simeq B^{mn}$ as R-modules.

$\Rightarrow \mathrm{Hom}_R(A,B)$ is an R-submodule of B^{mn}. 65

B^{mn} fin. gen R-module $+$ R noetherian

$\Rightarrow \mathrm{Hom}_R(A,B)$ fin. gen. R-module.

(b) $\mathrm{End}_\Lambda(A) = \mathrm{Hom}_\Lambda(A,A)$ - fin. gen. R-module by (a).

R-algebra structure is given by

$r \longmapsto r \cdot 1_A$

(c) Λ fin. gen R-module $\Rightarrow \ell_R(\Lambda) < \infty$.

$\Rightarrow \Lambda$ artinian (and noetherian) R-module.

Every left ideal in Λ is an R-submodule of Λ

$\Rightarrow \Lambda$ is left artinian. $\square$

Categories and functors.

DEF: A category $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}\,\mathcal{C}$ and for each pair (A,B) in $\mathcal{C}$ a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$ (can be $\emptyset$), write $f: A \to B$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, and composition of morphisms

$$\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$
$$(g,f) \longmapsto gf$$

such that

(i) for each object A in $\mathcal{C}$, $\exists\, 1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ such that $f \cdot 1_A = f$ $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $1_A \cdot g = g$ $\forall g \in \mathrm{Hom}_{\mathcal{C}}(C,A)$.

(ii) associative law is satisfied

$$h(gf) = (hg)f$$

when $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$.

Examples

1) Γ quiver, $J^t \subseteq \langle \rho \rangle \subseteq J^2$, $t \geq 2$.

$\mathrm{Rep}(\Gamma,\rho)$ = category of all representations of (Γ,ρ)

$\mathrm{Obj}(\mathrm{Rep}(\Gamma,\rho))$ = all representations of (Γ,ρ) over k.

morphisms = morphisms of representations

composition = composition of morphisms of representations.

2) Λ ring

$\mathrm{Mod}\,\Lambda$ = the category of all left Λ-modules.

$\mathrm{Obj}(\mathrm{Mod}\,\Lambda)$ = all left Λ-modules.

morphisms = Λ-homomorphisms of left Λ-modules

composition = usual, composition of maps

Special cases: Ab = abelian grps = $\mathrm{Mod}\,\mathbb{Z}$.

$\mathrm{Vec}(k)$ = vector spaces over k = $\mathrm{Mod}\,k$.

DEF: $\mathcal{C}$ category, A, B objects in $\mathcal{C}$. A morphism $f: A \to B$ in $\mathcal{C}$ is an isomorphism in $\mathcal{C}$ if $\exists$ a morphism $g: B \to A$ in $\mathcal{C}$ such that

$$gf = 1_A \quad \text{and} \quad fg = 1_B.$$

Note: Λ ring, $f: A \to B$ Λ-hom.

f isomorphism $\overset{\text{def}}{\Longleftrightarrow}$ f is bijective.

$\Longleftrightarrow$ f is an isomorphism in $\mathrm{Mod}\,\Lambda$

DEF: $\mathcal{C}$ category. A category $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if $\mathrm{Obj}\,\mathcal{D} \subseteq \mathrm{Obj}\,\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{D}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$$

for all $A,B \in \mathcal{D}$, and the composition in $\mathcal{D}$ is the restriction of the composition in $\mathcal{C}$.

$\mathcal{D}$ is a full subcategory of $\mathcal{C}$ if

$$\mathrm{Hom}_{\mathcal{D}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$$

for $A,B \in \mathcal{D}$.

Note: Full subcategory - enough to describe the objects in the subcategory.

Examples

① Λ not commutative: $\mathrm{Mod}\,\Lambda$ is a subcategory of $\mathrm{Mod}\,\mathbb{Z} = Ab$, which is not full:

$\mathrm{Hom}_\Lambda(\Lambda,\Lambda) \subsetneq \mathrm{Hom}_\mathbb{Z}(\Lambda,\Lambda)$:

Choose $z \in Z(\Lambda)$, $f: \Lambda \to \Lambda$ by $f(\lambda) = z\cdot\lambda$.

Then $f \in \mathrm{Hom}_\mathbb{Z}(\Lambda,\Lambda)$, but $f \notin \mathrm{Hom}_\Lambda(\Lambda,\Lambda)$.

② Λ ring, $I \subseteq \Lambda$ ideal, $\pi: \Lambda \to \Lambda/I$ natural proj.

Any Λ/I-module M is also a Λ-module via $\lambda\cdot m \overset{def}{=} \pi(\lambda)m$

Exercise: $\mathrm{Mod}\,\Lambda/I \subseteq \mathrm{Mod}\,\Lambda$ is a full subcat.

(Λ/I infinite rep. type $\Rightarrow \Lambda$ infinite rep type)

③ Λ ring, $M \in \mathrm{Mod}\,\Lambda$

$\mathrm{add}\,M$ = all direct summands in a finite number of copies of M

($X \in \mathrm{add}\,M$; $M^n = X \oplus Y$, $n<\infty$).

DEF: A (covariant) functor $F: \mathcal{C} \to \mathcal{D}$ associates to each object C in $\mathcal{C}$ an object $F(C)$ in $\mathcal{D}$, and to each morphism $f: A \to B$ in $\mathcal{C}$ a morphism $F(f): F(A) \to F(B)$ in $\mathcal{D}$ such that

(i) $F(gf) = F(g)F(f)$ all composable morphisms in $\mathcal{C}$,

(ii) $F(1_A) = 1_{F(A)}$, $\forall A \in \mathrm{Obj}(\mathcal{C})$.

F contravariant functor:

$f: A \to B \rightsquigarrow F(f): F(B) \to F(A)$

(i) $F(gf) = F(f)F(g)$

(ii) $F(1_A) = 1_{F(A)}$

Examples

(1) $\Lambda = k\Gamma/\langle\rho\rangle$, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1,2,\dots,n\}$

$F : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{Rep}(\Gamma,\rho)$

$M \longmapsto F(M) = (V,f)$

$V(i) = \bar{e}_i M$

$\alpha : i \to j \in \Gamma_1$.

$f_\alpha : V(i) = \bar{e}_i M \xrightarrow{\bar{\alpha}\cdot -} \bar{e}_j M = V(j)$

$\bar{e}_i m \longmapsto \bar{\alpha}\cdot\bar{e}_i m$

$$\begin{tikzcd} M \arrow[d, "h"] & \longmapsto & F(M) = (V,f) \arrow[d, "F(h)"] & V(i) = \bar{e}_i M \arrow[d, "F(h)(i) = h|_{\bar{e}_i M}"] \\ M' & & F(M') = (V',f') & V'(i) = \bar{e}_i M' \end{tikzcd}$$

(2) $\mathcal{C}$ category, $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$

$id_{\mathcal{C}}(C) = C$, $\forall C \in \mathrm{Obj}(\mathcal{C})$

$f : A \to B$ in $\mathcal{C}$

$id_{\mathcal{C}}(f) = f : id_{\mathcal{C}}(A) = A \xrightarrow{f} B = id_{\mathcal{C}}(B)$

$id_{\mathcal{C}}$ = identity functor.

(3) $A \in \mathrm{Mod}\,\Lambda$

$F = \mathrm{Hom}_\Lambda(A,-) : \mathrm{Mod}\,\Lambda \longrightarrow \mathrm{Ab}$

$F(B) = \mathrm{Hom}_\Lambda(A,B)$

$f : B \to C$, $F(f) : \underset{F(B)}{\mathrm{Hom}_\Lambda(A,B)} \longrightarrow \underset{F(C)}{\mathrm{Hom}_\Lambda(A,C)}$

$g : A \to B \longmapsto F(g) = f\cdot g$

(4) Γ quiver

$\mathrm{Obj}(\Gamma) = \Gamma_0$

morphisms $i \to j$ = all paths from i to j.

$F : \Gamma \longrightarrow \mathrm{mod}\,k = \mathrm{vec}(k)$

$\rightsquigarrow$ representation of Γ over k.

<u>DEF</u>: $\mathcal{C}$ category, R comm. ring. $\mathcal{C}$ is <u>preadditive</u> (R-category) if $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an abelian group (R-module) for all objects A and B in $\mathcal{C}$ and the composition

$$\varphi: \mathrm{Hom}_{\mathcal{C}}(B,C)\times \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$

is bilinear (R-bilinear) for all A, B and C in $\mathcal{C}$, i.e.

$$\varphi(g_1+g_2, f) = \varphi(g_1,f)+\varphi(g_2,f)$$
$$\varphi(g, f_1+f_2) = \varphi(g,f_1)+\varphi(g,f_2)$$

$$\Big(\varphi(g, rf) = \varphi(rg, f) = r\varphi(g,f) \quad R\text{-bilinear}\Big)$$

<u>DEF</u>: $\mathcal{C}, \mathcal{D}$ preadditive (R-) categories. A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is an <u>additive</u> (R-) functor if the map

$$F: \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is a homomorphism of groups (R-modules) for all objects A and B in $\mathcal{C}$.

<u>Examples</u>

(1) $\Lambda = k\Gamma/\langle\rho\rangle$, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1,2,\dots,n\}$

$\mathrm{Rep}(\Gamma,\rho)$ – preadditive k-category.

$\mathrm{mod}\,\Lambda$ – ——— " ———

$F: \mathrm{mod}\,\Lambda \longrightarrow \mathrm{Rep}(\Gamma,\rho)$ add. k-functor

$H: \mathrm{Rep}(\Gamma,\rho) \longrightarrow \mathrm{mod}\,\Lambda$ add. —— " ——

$$(V,f) \longmapsto H(V,f) = V(1)\oplus V(2)\oplus\cdots\oplus V(n)$$

$$\bar{e}_i \cdot (v_1, v_2, \dots, v_n) = (0,\dots,0,v_i,0,\dots,0)$$

i-th coord.

$\alpha: i \to j \in \Gamma_1$

$$\bar{\alpha}(v_1, v_2, \dots, v_n) = (0,\dots,0,f_\alpha(v_i),0,\dots,0)$$

$$\begin{array}{c} (V,f) \\ \downarrow h \\ (V',f') \end{array} \qquad H(h) = h_1\oplus h_2\oplus\cdots\oplus h_n : H(V,f) = V(1)\oplus V(2)\oplus\cdots\oplus V(n) \to H(V',f') = V'(1)\oplus V'(2)\oplus\cdots\oplus V'(n)$$

(2) $\mathcal{C}$ preadditive, $\mathcal{D} \subseteq \mathcal{C}$ full subcategory
$\Rightarrow \mathcal{D}$ ~~pre~~preadditive

In particular, Λ artin R-algebra
$\mathrm{Mod}\Lambda$, $\mathrm{mod}\Lambda$, $\mathrm{add}\, M$ are R-category

$\mathrm{Hom}_\Lambda(A,-) : \mathrm{mod}\Lambda \longrightarrow \mathrm{mod} R$
additive R-functor.

<u>Morphisms of functors</u>

<u>DEF:</u> $\mathcal{C}, \mathcal{D}$ categories, $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors. Then $\varphi : F \longrightarrow G$ is a <u>morphism</u> if for each object $C \in \mathcal{C}$ there is a morphism $\varphi_C : F(C) \longrightarrow G(C)$ in $\mathcal{D}$ such that for each morphism $f : A \longrightarrow B$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & \circlearrowright & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\varphi_B} & G(B) \end{array}$$

$\varphi : F \longrightarrow G$ is an <u>isomorphism</u> if $\varphi_C : F(C) \longrightarrow G(C)$ is an isomorphism in $\mathcal{D}$ for all C in $\mathcal{C}$.

<u>Examples</u> (1) k field

$(\ )^* = \mathrm{Hom}_k(-,k) : \mathrm{Vec}(k) \to \mathrm{Vec}(k)$

Define $\varphi : \mathrm{id}_{\mathrm{Vec}(k)} \to (\ )^{**} = \mathrm{Hom}_k(\mathrm{Hom}_k(-,k),k)$ by

$$\varphi_V : \mathrm{id}_{\mathrm{Vec}(k)}(V) = V \longrightarrow V^{**} = \mathrm{Hom}_k(\mathrm{Hom}_k(V,k),k)$$
$$v \longmapsto \varphi_V(v), \text{ where}$$
$$\varphi_V(v)(f) = f(v), \quad f \in \mathrm{Hom}_k(V,k)$$

φ is a morphism of functors.

<u>Exercise:</u> (i) φ_V is 1-1
(ii) $\dim_k V < \infty \to \varphi_V$ is an isom.

(2) Γ quiver, $F, G : \Gamma \to \mathrm{vec}(k)$ functors,
What is a morphism $\varphi : F \longrightarrow G$?

<u>DEF:</u> $\mathcal{C}, \mathcal{D}$ (R-) categories, $F: \mathcal{C} \to \mathcal{D}$ an (R-) functor. Then $F: \mathcal{C} \to \mathcal{D}$ is an <u>equivalence</u> of (R-) categories, if $\exists$ (R-) functor $H: \mathcal{D} \to \mathcal{C}$ such that

$$HF \simeq id_{\mathcal{C}} \quad \text{and} \quad FH \simeq id_{\mathcal{D}}.$$

<u>DEF:</u> $F: \mathcal{C} \to \mathcal{D}$ (R-) functor

(a) F is <u>full</u>, if

$$F: Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(F(A), F(B))$$

is onto for all A, B in $\mathcal{C}$.

(b) F is <u>faithful</u>, if

$$F: Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(F(A), F(B))$$

is 1-1 for all A, B in $\mathcal{C}$.

(c) F is <u>dense</u>, if for each object D in $\mathcal{D}$, $\exists$ an object C in $\mathcal{C}$ such that $F(C) \simeq D$.

<u>Can be shown:</u>

<u>Proposition 45</u> $\mathcal{C}, \mathcal{D}$ (R-) categories, $F: \mathcal{C} \to \mathcal{D}$ (R-) functor

F is an equivalence

$\iff F$ is full, faithful and dense.

<u>Theorem 46</u>

(Γ, ρ) quiver with relations, $J^t \subseteq \langle \rho \rangle \subseteq J^2$, $\Lambda = k\Gamma/\langle \rho \rangle$. The functors $F: \text{mod}\Lambda \to Rep(\Gamma, \rho)$ and $H: Rep(\Gamma, \rho) \to \text{mod}\Lambda$ are inverse equivalences of k-categories,

$$HF \simeq id_{\text{mod}\Lambda} \quad \text{and} \quad FH \simeq id_{Rep(\Gamma,\rho)}.$$

<u>Proof:</u> (1) <u>F equivalence:</u>

(i) <u>F faithful:</u> $h: A \to B \in \text{mod}\Lambda$, $\Gamma_0 = \{1, 2, \dots, n\}$

$F(h) = \{h_i\}_{i=1}^n$, where $h_i = h|_{\bar{e}_i A}: \bar{e}_i A \to \bar{e}_i B$

$F(h) = 0 \Rightarrow h_i = 0, \ \forall i = 1, 2, \dots, n$

$h_i = h|_{\bar{e}_i A} = 0$

$\Rightarrow h = \underbrace{h|_{\bar{e}_1 A}}_{=0} \oplus \underbrace{h|_{\bar{e}_2 A}}_{=0} \oplus \dots \oplus \underbrace{h|_{\bar{e}_n A}}_{=0} : \bar{e}_1 A \oplus \dots \oplus \bar{e}_n A \to B$

$\Rightarrow h = 0 \Rightarrow F$ is faithful.

(ii) <u>F dense</u>:

<u>Check</u>: Given $(V,f) \in \mathrm{Rep}(\Gamma,\rho)$, then

$$F\,H(V,f) \simeq (V,f), \quad \forall (V,f) \in \mathrm{Rep}(\Gamma,\rho)$$

$\Rightarrow$ F dense.

(iii) <u>F full</u>: Given $\{h_i\} : F(A) \longrightarrow F(B)$

i.e. $h_i : \bar{e}_i A \longrightarrow \bar{e}_i B$ for $i = 1, 2, \dots, n$

<u>Know</u>: $\tilde{h} = h_1 \oplus h_2 \oplus \cdots \oplus h_n : A \longrightarrow B \in \mathrm{Hom}_R(A,B)$

$$A = \bar{e}_1 A \oplus \bar{e}_2 A \oplus \cdots \oplus \bar{e}_n B$$

<u>Check</u>: $F(\tilde{h}) = \{h_i\} \Rightarrow F$ full.

$\Rightarrow$ F is an equivalence.

(2) <u>F and H are inverse equivalences</u>:

Exercise. $\square$

<u>Can be shown</u>:

$F : \mathrm{Mod}\,R \longrightarrow \mathrm{Mod}\,S$ equivalence.

- M (semi)simple, fin. gen., artinian, noetherian, indec.

$\Longleftrightarrow$ $F(M)$ is

- $Z(R) \simeq Z(S)$.

Projectivization

Λ artin R-algebra, $A \in \operatorname{mod}\Lambda$

Have seen: $\Gamma = \operatorname{End}_\Lambda(A)^{op}$ is an artin R-alg.

A is a left $\operatorname{End}_\Lambda(A)$-module:

$f \in \operatorname{End}_\Lambda(A)$, $a \in A$, then

$$f \cdot a \overset{\text{def}}{=} f(a) \qquad \text{(Exercise)}$$

$\Rightarrow A$ is a right Γ-module.

Exercise: • ${}_\Lambda A_\Gamma$ is a Λ-Γ-bimodule

- left Λ-module
- right Γ-module
- $(\lambda \cdot a)\gamma = \lambda(a\gamma)$

• $\operatorname{Hom}_\Lambda({}_\Lambda A_\Gamma, {}_\Lambda X)$ – left Γ-module, ($X \in \operatorname{mod}\Lambda$)

$f \in$; $g \in \Gamma$; $(g \cdot f)(a) \overset{\text{def}}{=} f(g(a))$, $\forall a \in A$

$\Rightarrow$ Have R-functor

$$F = \operatorname{Hom}_\Lambda(A,-) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$$

R-categories.

Lemma 47 Λ ring

(a) $\operatorname{Hom}_\Lambda(A, B_1 \oplus B_2) \overset{\alpha}{\leftarrow} \operatorname{Hom}_\Lambda(A,B_1) \oplus \operatorname{Hom}_\Lambda(A,B_2)$ given by $\alpha(f,g)(a) \overset{\text{def}}{=} (f(a), g(a))$ for $a \in A$, is an isomorphism.

(b) $\operatorname{Hom}_\Lambda(A_1 \oplus A_2, B) \simeq \operatorname{Hom}_\Lambda(A_1,B) \oplus \operatorname{Hom}_\Lambda(A_2,B)$

In particular, if $\Gamma = \operatorname{End}_\Lambda(A)^{op}$, then the isomorphism α in (a) is an isomorphism of Γ-modules. Similarly in (b).

$\square$

Proof: Exercise.

Proposition 48

Λ artin R-algebra, $A \in \operatorname{mod}\Lambda$, $\Gamma = \operatorname{End}_\Lambda(A)^{op}$

$e_A = \operatorname{Hom}_\Lambda(A,-) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ (R-functor) has the following properties:

(a) $e_A : \operatorname{Hom}_\Lambda(Z,X) \xrightarrow{\sim} \operatorname{Hom}_\Gamma(e_A(Z), e_A(X))$

is an R-isomorphism for all Z in $\operatorname{add} A$ (i.e. $\exists Y$ such that $Z \oplus Y \simeq A^t$ for some $t \geq 1$)

(b) $X \in \operatorname{add} A \Rightarrow e_A(X)$ is a projective Γ-module.

(c) $e_A|_{\operatorname{add} A} : \operatorname{add} A \longrightarrow \mathcal{P}(\Gamma) = \{\text{fin. gen. projective } \Gamma\text{-mod's}\}$

is an equivalence of R-categories.

<u>Proof</u>: (a) (i) $e_A : \operatorname{Hom}_\Lambda(A,X) \xrightarrow{\alpha} \operatorname{Hom}_\Gamma(e_A(A), e_A(X))$

is an isomorphism of R-modules. $\quad f \longmapsto \alpha(f) = f_* : {}_\Lambda(A,A) \to {}_\Lambda(A,X)$, $\quad g \longmapsto f \cdot g$

- α is an R-homomorphism : Exercise.
- <u>α 1-1</u>: Assume that $\alpha(f) = f_* = 0$
  $\Longrightarrow f_*(g) = f \cdot g = 0, \ \forall g \in \operatorname{Hom}_\Lambda(A,A)$.
  Choose $g = 1_A \Longrightarrow f = 0 \Longrightarrow \alpha$ is 1-1.
- <u>α onto</u>: Given $h : \operatorname{Hom}_\Lambda(A,A) \to \operatorname{Hom}_\Lambda(A,X)$

Let $f = h(1_A) : A \to X \in \operatorname{Hom}_\Lambda(A,X)$, Then for $g \in \operatorname{Hom}_\Lambda(A,A)$,

action of Γ on $\operatorname{Hom}_\Lambda(A,X)$

$$\alpha(f)(g) = fg = h(1_A) \cdot g \ \cancel{\neq g(h(1_A))} = h(1_A \cdot g) = h(g)$$

h Γ-hom.

$\Longrightarrow \alpha(f) = h \Longrightarrow \alpha$ is onto.

(ii) <u>e_A "additive"</u> : The following diagram commutes, $i_M : M \longrightarrow M \oplus N \xleftarrow{i_N} N$

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(M \oplus N, X) \arrow[r, "e_A"] \arrow[dd, "\wr"] & \operatorname{Hom}_\Gamma(e_A(M \oplus N), e_A(X)) \arrow[d, "\wr", "\text{Lemma 47 a)}"'] \\
& \operatorname{Hom}_\Gamma(e_A(M) \oplus e_A(N), e_A(X)) \arrow[d, "\wr"] \\
\operatorname{Hom}_\Lambda(M,X) \oplus \operatorname{Hom}_\Lambda(N,X) \arrow[r, "e_A \oplus e_A"] & \operatorname{Hom}_\Gamma(e_A(M), e_A(X)) \oplus \operatorname{Hom}_\Gamma(e_A(N), e_A(X))
\end{tikzcd}$$

$f \longmapsto f_*$

$(f i_M, f i_N) \longmapsto ((f i_M)_*, (f i_N)_*)$

$(f i_M)_* \oplus (f i_N)_*$

(iii) <u>$Z = A^n$</u>:

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(A^n, X) \arrow[r, "e_A"] \arrow[d, "\wr"] & \operatorname{Hom}_\Gamma(e_A(A^n), e_A(X)) \arrow[d, "\wr"] \\
\operatorname{Hom}_\Lambda(A,X)^n \arrow[r, "\overbrace{e_A \oplus \cdots \oplus e_A}^{n}"] & \operatorname{Hom}_\Gamma(e_A(A), e_A(X))^n
\end{tikzcd}$$

Commutes by (ii)

isomorphism by (i)

$\Longrightarrow e_A$ is an isomorphism for $Z = A^n$

(iv) $Z \in \operatorname{add} A$, $Z \oplus Y \simeq A^n$, $n \geq 1$: (same by (iii))

$$\begin{array}{ccc} \operatorname{Hom}_\Lambda(Z\oplus Y, X) & \xrightarrow{e_A} & \operatorname{Hom}_\Gamma(e_A(Z\oplus Y), e_A(X)) \\ \wr\| & \circlearrowright \text{ comm. by step (ii)} & \downarrow\wr \\ \operatorname{Hom}_\Lambda(Z,X)\oplus \operatorname{Hom}_\Lambda(Y,X) & \xrightarrow{e_A\oplus e_A} & \operatorname{Hom}_\Gamma(e_A(Z), e_A(X))\oplus \operatorname{Hom}_\Gamma(e_A(Y), e_A(X)) \end{array}$$

$\Rightarrow e_A \oplus e_A$ is an isomorphism

$\Rightarrow e_A : \operatorname{Hom}_\Lambda(Z,X) \to \operatorname{Hom}_\Gamma(e_A(Z), e_A(X))$ is an isomorphism for all $Z \in \operatorname{add} A$ and $X \in \operatorname{mod}\Lambda$.

(b) Let $X \in \operatorname{add} A$, i.e. $\exists n \geq 1$ such that $A^n \simeq X \oplus Y$ for some Y.

$\Longrightarrow e_A(A^n) = \operatorname{Hom}_\Lambda(A, A^n) \simeq \operatorname{Hom}_\Lambda(A,A)^n \simeq \Gamma^n$

$\wr\|$

$e_A(X\oplus Y) = \operatorname{Hom}_\Lambda(A, X\oplus Y)$

$\simeq \underbrace{\operatorname{Hom}_\Lambda(A,X)}_{e_A(X)} \oplus \operatorname{Hom}_\Lambda(A,Y)$

as Γ-modules

$\Rightarrow e_A(X)$ is a projective Γ-module.

(c) (b) $\Rightarrow e_A : \operatorname{add} A \longrightarrow \mathcal{P}(\Gamma)$

(a) $\Rightarrow e_A$ is full and faithful.

<u>e_A dense:</u> Let $P \in \mathcal{P}(\Gamma)$. Then $\exists n \geq 1$ and Q such that

$$\Gamma^n \overset{\varphi}{\simeq} P \oplus Q$$

Let $f' : P\oplus Q \longrightarrow P \oplus Q$ be given by $f'(p,q) = (0,q)$. Then $(f')^2 = f'$ and $\operatorname{Ker} f' \simeq P$, and

$$e_A : \operatorname{Hom}_\Lambda(A^n, A^n) \xrightarrow{\sim} \operatorname{Hom}_\Gamma(e_A(A^n), e_A(A^n)) \quad g$$

$$\xrightarrow[\sim]{\psi} \operatorname{Hom}_\Gamma(\Gamma^n, \Gamma^n) \quad \sigma g \sigma^{-1}$$

$$\varphi^{-1} f' \varphi$$

($\Gamma^n \overset{\sigma}{\simeq} e_A(A^n)$)

Choose $u : A^n \longrightarrow A^n$ such that

$$\psi\, e_A(u) = \varphi^{-1} f' \varphi : \Gamma^n \longrightarrow \Gamma^n$$

Let $f = \varphi^{-1} f' \varphi$. Note that $f^2 = f$. Have an exact sequence

$$0 \to \operatorname{Ker} u \longrightarrow A^n \xrightarrow{\ u\ } A^n$$

This induces the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & e_A(\mathrm{Ker}\,u) \arrow[r] & e_A(A^n) \arrow[r, "e_A(u)"] \arrow[dr, phantom, "\circlearrowright"] & e_A(A^n) \\
0 \arrow[r] & \mathrm{Ker}\,f \arrow[r] \arrow[u, dashed, "\wr"] \arrow[d, dashed, "\wr"] \arrow[ur, phantom, "\Leftarrow"] \arrow[dr, phantom, "\Leftarrow"] & \Gamma^n \arrow[r, "\psi e_A(u) = f"] \arrow[u, "\sigma", "\wr"'] \arrow[d, "\wr \varphi"] \arrow[dr, phantom, "\circlearrowright"] & \Gamma^n \arrow[u, "\sigma", "\wr"'] \arrow[d, "\wr \varphi"] \\
0 \arrow[r] & P \arrow[r] & P \oplus Q \arrow[r, "f'"] & P \oplus Q
\end{tikzcd}$$

Problem sheets $\Rightarrow e_A(\mathrm{Ker}\,u) \simeq \mathrm{Ker} f \simeq P$

<u>$\mathrm{Ker}\,u \in \mathrm{add}\,A$</u>: $f = \psi e_A(u) = f^2 = (\psi e_A(u))^2$

$$= \psi(e_A(u)^2) = \psi e_A(u^2)$$

$\Rightarrow u = u^2$, since ψ is an isomorphism and e_A is full and faithful. ($A^n \in \mathrm{add}\,A$)

$\Rightarrow A^n = \mathrm{Ker}\,u \oplus \mathrm{Im}\,u \Rightarrow \mathrm{Ker}\,u \in \mathrm{add}\,A$

$\Rightarrow P \in e_A(\mathrm{add}\,A) \Rightarrow e_A|_{\mathrm{add}A}$ is dense.

$\square$

<u>Lemma 49</u>

Λ artin R-algebra, $A \in \mathrm{mod}\,\Lambda$, $\Gamma = \mathrm{End}_\Lambda(A)^{op}$

$$e_A : \mathrm{add}\,A \longrightarrow \mathcal{P}(\Gamma)$$

(a) $X \neq (0)$ in $\mathrm{add}\,A \Leftrightarrow e_A(X) \neq (0)$ in $\mathcal{P}(\Gamma)$

For $X \in \mathrm{add}\,A$.

(b) $X \in \mathrm{add}\,A$

X indecomposable $\Leftrightarrow e_A(X)$ indecomposable

(c) $X, Y \in \mathrm{add}\,A$: $e_A(X) \simeq e_A(Y) \Leftrightarrow X \simeq Y$.

<u>Proof</u>: (a) Prop 48 $\Rightarrow \mathrm{Hom}_\Lambda(X, X) \simeq \mathrm{Hom}_\Gamma(e_A(X), e_A(X))$ for $X \in \mathrm{add}\,A$.

<u>$\Rightarrow$</u>: $e_A(X) = (0) \Rightarrow \mathrm{Hom}_\Lambda(X, X) = 0 \Rightarrow 1_X = 0 \Rightarrow X = (0)$.

<u>$\Leftarrow$</u>: $X = (0) \Rightarrow e_A(X) = \mathrm{Hom}_\Lambda(A, (0)) = (0)$.

(b) Have: $\mathrm{End}_\Lambda(X) \simeq \mathrm{End}_\Gamma(e_A(X))$ as rings.
Follows from this.

(c) <u>Exercise</u>: General property of an equivalence

$\square$