Basic artin algebras

DEF: Λ artin R-algebra. Then Λ is *basic* if $\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ with P_i indecomposable, then $P_i \not\simeq P_j$ for $i \neq j$.

Recall: $\Lambda/\underline{r} = S_1 \oplus \cdots \oplus S_n$ - semisimple, S_i simple.

$P(S_i) \longrightarrow S_i$ proj. cover $\Rightarrow \Lambda \simeq P(S_1) \oplus \cdots \oplus P_n(S)$

Have: $P(S_i) \simeq P(S_j) \iff S_i \simeq S_j$

Λ basic $\iff S_i \not\simeq S_j$ for $i \neq j$.

Examples

(1) $\Lambda = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$, k field.

$\Lambda = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$

simple Λ-modules $\Rightarrow$ indec.

$\begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \xrightarrow{-\cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$ - isomorphism of left Λ-mod's

$\Rightarrow \Lambda$ is not basic.

(2) (Γ, ρ) quiver with relations$/k$

$J^t \leq \langle \rho \rangle \leq J^2$, $\Gamma_0 = \{1, 2, \ldots, n\}$.

$\Lambda = k\Gamma/\langle \rho \rangle$ is basic:

$\Lambda/\underline{r} \simeq k\bar{e}_1 \oplus k\bar{e}_2 \oplus \cdots \oplus k\bar{e}_n$ with $k\bar{e}_i \not\simeq k\bar{e}_j$ for $i \neq j$.

$\Rightarrow \Lambda$ is basic.

Note: Λ artin R-algebra

Know: $\Lambda \simeq P_1^{n_1} \oplus P_2^{n_2} \oplus \cdots \oplus P_t^{n_t}$, P_i indec. $P_i \not\simeq P_j$ for $i \neq j$.

Λ basic $\iff n_i = 1$ for all $i = 1, 2, \ldots, t$.

Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_t$, and let $\Sigma = \operatorname{End}_\Lambda(P)^{op}$.

Have that $e_P = \operatorname{Hom}_\Lambda(P, -) : \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Sigma$

where

$e_P(P) = \Sigma \simeq e_P(P_1) \oplus e_P(P_2) \oplus \cdots \oplus e_P(P_t)$

indec, since P_i is indec

$e_P(P_i) \not\simeq e_P(P_j)$ for $i \neq j$, since $P_i \not\simeq P_j$ for $i \neq j$ $\Rightarrow \Sigma$ is basic.

<u>Note</u>: If Λ is basic, then $P = \Lambda$ and

$$\Sigma = \mathrm{End}_\Lambda(\Lambda)^{op} \simeq \Lambda.$$

<u>Proposition 49</u> Λ and Σ be as above.

Then
$$e_P = \mathrm{Hom}_\Lambda(P,-) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Sigma$$
is an equivalence of k-categories.

<u>Proof</u>: <u>Know</u>: e_P is an k-functor.

Prop. 48(c) $\Rightarrow$ $e_P|_{\mathrm{add}P} : \mathrm{add}P \to \mathcal{P}(\Sigma)$ is an equivalence.

<u>e_P dense</u>: Let $C \in \mathrm{mod}\,\Sigma$. Then $\exists$ exact sequence $Q_1 \xrightarrow{f} Q_0 \longrightarrow C \to 0$ with $Q_0, Q_1 \in \mathcal{P}(\Sigma)$. Since $Q_1 \xrightarrow{f} Q_0 \in \mathcal{P}(\Sigma)$, $\exists\, Q_1', Q_0' \in \mathrm{add}P$ and $f' : Q_1' \to Q_0' \in \mathrm{add}P$, such that $e_P(Q_i') \simeq Q_i$ and

$$\underbrace{\begin{array}{ccccc} e_P(Q_1') & \xrightarrow{e_P(f')} & e_P(Q_0') & \longrightarrow & e_P(\mathrm{Coker} f') \to 0 \\ \wr\wr & \circlearrowleft & \wr\wr & \Longrightarrow & \downarrow \wr\wr \\ Q_1 & \xrightarrow{f} & Q_0 & \longrightarrow & C \to 0 \end{array}}_{\text{is commutative}}$$

$Q_1' \xrightarrow{f'} Q_0' \longrightarrow \mathrm{Coker} f' \to 0$ exact

<u>Exercise</u>: P projective $\Longrightarrow$
$$e_P(Q_1') \xrightarrow{e_P(f')} e_P(Q_0') \to e_P(\mathrm{Coker} f') \to 0$$
exact.

Problem sheets $\Rightarrow$ $e_P(\mathrm{Coker} f') \simeq C$.

$\Rightarrow$ e_P is dense.

<u>e_P full and faithful</u>: Let $X, Y \in \mathrm{mod}\,\Lambda$, $\exists$

$\eta' : Q_1' \longrightarrow Q_0' \longrightarrow X \longrightarrow 0$ exact in $\mathrm{mod}\,\Lambda$

with $Q_i' \in \mathrm{add}P = \mathcal{P}(\Lambda)$.

$\Rightarrow \eta : e_P(Q_1') \longrightarrow e_P(Q_0') \longrightarrow e_P(X) \longrightarrow 0$ exact.

Apply $\mathrm{Hom}_\Lambda(-,Y)$ to η':

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_\Lambda(X,Y) & \longrightarrow & \mathrm{Hom}_\Lambda(Q_0',Y) & \xrightarrow{\text{Prop }48(c)} & \mathrm{Hom}_\Lambda(Q_1',Y) \\ \text{Problem sheets} \} \text{exact} & e_P \downarrow \wr\wr & \Leftarrow & e_P\, \wr\wr & \circlearrowleft & e_P\, \wr\wr \\ 0 \to & \mathrm{Hom}_\Sigma(e_P(X), e_P(Y)) & \to & \mathrm{Hom}_\Sigma(e_P(Q_0'), e_P(Y)) & \to & \mathrm{Hom}_\Sigma(e_P(Q_1'), e_P(Y)) \end{array}$$

$\Rightarrow e_P$ full and faithful $\Rightarrow e_P : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Sigma$ equivalence $\square$

<u>DEF</u>: Λ, Σ two rings. Then Λ and Σ are <u>Morita equivalent</u> if $\mathrm{Mod}\,\Lambda$ and $\mathrm{Mod}\,\Sigma$ are equivalent categories.

(Anderson & Fuller p. 266 Exer. 4): (or $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Sigma$ are equivalent categories)

Theorem 50

Let Λ be a fin. dim. k-algebra over an algebraically closed field k.

(a) $\exists$ a basic k-algebra Σ such that Λ and Σ are Morita equivalent.

(b) Suppose that Λ is basic. Then $\exists (\Gamma, \rho)$ a quiver with relations over k, such that $\Lambda \simeq k\Gamma/\langle\rho\rangle$, where $J^t \subseteq \langle\rho\rangle \subseteq J^2$ for some $t \geq 2$.

Proof: (a) Prop. 49.

(b) ① Claim: $\Lambda/\underline{r} \simeq k^s$ for some $s \geq 1$.

Know: $\Lambda/\underline{r}$ semisimple $\Rightarrow \Lambda/\underline{r} \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$

D_i division ring.

$$e_{11}^{(1)} = \mathrm{diag}(1,0,\ldots,0) = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}, \quad e_{22}^{(1)} = \mathrm{diag}(0,1,0,\ldots,0) \in M_{n_1}(D_1)$$
$$e_{n_1 n_1}^{(1)} = (0,\ldots,0,1)$$

– complete set of primitive orthogonal idempotents of $M_{n_1}(D_1)$.

Similarly for the other $M_{n_i}(D_i)$. Let

$$f_{ij} = (0,\ldots,0,\underset{\text{i-th coordinate}}{e_{jj}^{(i)}},0,\ldots,0), \quad j = 1,2,\ldots,n_i$$

$\Rightarrow \bigcup_{i=1}^{r} \{f_{ij}\}_{j=1}^{n_i}$ complete set of primitive orthogonal idempotents for $\Lambda/\underline{r}$.

Have $\Lambda/\underline{r} f_{ij} \simeq \Lambda/\underline{r} f_{ij'}$ $\forall i$, $\forall j,j' \in \{1,2,\ldots,n_i\}$.

Lift $\bigcup_{i=1}^{r}\{f_{ij}\}_{j=1}^{n_i}$ to Λ; $\bigcup_{i=1}^{r}\{\tilde{f}_{ij}\}_{j=1}^{n_i}$, a complete set of primitive orthogonal idempotents in Λ.

Know: $\Lambda\tilde{f}_{ij} \longrightarrow \Lambda\tilde{f}_{ij}/\underline{r}\tilde{f}_{ij} \simeq \Lambda/\underline{r} f_{ij}$ proj. cover

$\Longrightarrow \Lambda\tilde{f}_{ij} \simeq \Lambda\tilde{f}_{ij'}$, $\forall j,j' \in \{1,2,\ldots,n_i\}$

Λ basic $\Rightarrow n_i = 1$ for all $i = 1,2,\ldots,r$.

$\Longrightarrow \Lambda/\underline{r} \simeq D_1 \times D_2 \times \cdots \times D_r$

② Claim: $D_i \simeq k$ for all $i = 1,2,\ldots,r$, d_i

$\dim_k \Lambda < \infty \Rightarrow \dim_k \Lambda/\underline{r} < \infty \Rightarrow \dim_k D_i < \infty$ $\forall i = 1,2,\ldots,r$

Have: $k \xhookrightarrow{\nu} D_i$, $k \simeq \nu(k) \hookrightarrow D_i$

Suppose $D_i \setminus \nu(k) \neq \emptyset$. Let $z \in D_i \setminus \nu(k)$.

Then $\{1, z, z^2, \ldots, z^{d_i}\}$ is linearly dependent,

$\Rightarrow \exists\, a_i \in \nu(k)$, $i = 0, 1, 2, \dots, d_i$ such that

$$a_0 \cdot 1 + a_1 z + a_2 z^2 + \cdots + a_{d_i} z^{d_i} = 0$$

that is, z is a root in the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{d_i} x^{d_i}$$

$k \simeq \nu(k)$ - alg. closed $\Rightarrow z \in \nu(k)$ ⨳

$\Rightarrow D_i = \nu(k) \simeq k$ for all i.

$\Rightarrow \Lambda/\underline{r} \simeq k^r \ni f_i = (0, \dots, 0, \underset{i\text{th coord.}}{1}, 0, \dots, 0)$

complete set of primitive orthogonal idempotents

- Lift $\{f_i\}_{i=1}^r$ to a complete set of primitive orthogonal idempotents $\{v_i\}_{i=1}^r$ in Λ
- Choose a basis $B_{ji} = \{a_{ji}(l)\}_l$ of $v_j \, \underline{r}/\underline{r}^2 \, v_i$ for all $i, j \in \{1, 2, \dots, r\}$.
- Lift ~~the~~ the elements in each B_{ji} to elements in $v_j \underline{r} v_i$, say $\tilde{B}_{ji} = \{\tilde{a}_{ji}(l)\}_l$
- Define Γ, $\Gamma_0 = \{i\}_{i=1}^r$

$$\Gamma_1 = \left\{ \alpha_{ji}(l) : i \to j \;\middle|\; l = 1, 2, \dots, \dim_k v_j \underline{r}/\underline{r}\, v_i \right\}$$

- Define $\varphi : k\Gamma \longrightarrow \Lambda$ by letting

$$\varphi(e_i) = v_i$$
$$\varphi(\alpha_{ji}(l)) = \tilde{a}_{ji}(l)$$

Problem sheets $\Rightarrow \varphi : k\Gamma \longrightarrow \Lambda$ is a k-alg. homomorphism

(80)

③ $\underline{\operatorname{Ker}\varphi \text{ admissible}}$: Suppose that

$$(*) \quad \varphi\Big(\underbrace{\sum_i \gamma_i(0) e_i + \sum_{r,s,l} \gamma_{rsl}(1) \alpha_{rs}(l) + \text{longer paths}}_{= x}\Big) = 0$$

i) $(*) \Longrightarrow \varphi(x) + \underline{r} = 0$

$$\parallel$$

$$\sum_i \gamma_i(0) v_i + \underbrace{\text{"}\langle \tilde{a}_{ji}(l) \mid j, i, l \rangle\text{"}}_{\subseteq \underline{r}} + \underline{r} = 0$$

$\Longrightarrow \sum_i \gamma_i(0) v_i + \underline{r} = 0$

$\Longrightarrow \sum_i \gamma_i(0) f_i = 0$ in $\Lambda/\underline{r} \simeq k^r$ $\Rightarrow \gamma_i(0) = 0 \quad \forall i = 1, 2, \dots, r$

$\{f_i\}$ basis for $\Lambda/\underline{r} \simeq k^r$

(ii) $(*) \Longrightarrow \varphi(x) + \underline{r}^2 = 0$

$$\parallel$$

$$\sum_{r,s,l} \gamma_{rsl}(1)\tilde{a}_{rs}(l) + \underbrace{\text{"}\langle \tilde{a}_{rs}(l)\tilde{a}_{r's'}(l')\rangle\text{"}}_{\subseteq \underline{r}^2} + \underline{r}^2$$

$$\Longrightarrow \sum_{r,s,l} \gamma_{rsl}(1)\tilde{a}_{rs}(l) + \underline{r}^2 = 0$$

$$\Longrightarrow \sum_{r,s,l} \gamma_{rsl}(1) a_{rs}(l) = 0 \qquad \underline{r}/\underline{r}^2.$$

basis for

$\Longrightarrow \gamma_{rsl}(1) = 0, \ \forall r,s,l \Longrightarrow x \in J^2$

$\Longrightarrow \operatorname{Ker}\varphi \subseteq J^2.$

<u>Have</u>: $\alpha_{rs}(l) \longmapsto \tilde{a}_{rs}(l) \in \underline{r}$

$\underline{r}^m = (0)$ for some $m \geq 1.$

$\Longrightarrow \{$all paths of length $\geq m\} \subseteq \operatorname{Ker}\varphi$

$\Longrightarrow J^m \subseteq \operatorname{Ker}\varphi$

$\Longrightarrow \operatorname{Ker}\varphi$ is admissible.

(4) <u>φ onto</u>: Let $\lambda \in \Lambda$. Then $\exists \gamma_i(0) \in k$ for $i = 1, 2, \ldots, r$ such that

$$x_1 = \lambda - \sum \gamma_i(0) v_i \in \underline{r},$$

since $\{\bar{v}_i = v_i + \underline{r}\}_{i=1}^{r}$ is a basis for $\Lambda/\underline{r}$

<u>Can show</u>: $\underline{r}$ is generated by $\{\tilde{a}_{ji}(l)\}_{j,i,l}$ and $\tilde{a}_{ji}(l)$ is in the image of φ. $\Longrightarrow \varphi$ is onto.

$\Longrightarrow k\Gamma/\operatorname{Ker}\varphi \simeq \operatorname{Im}\varphi = \Lambda \qquad \square$