

Duality

DEF: \mathcal{C}, \mathcal{D} (R -) categories, $F: \mathcal{C} \rightarrow \mathcal{D}$
 ~~R -) functor~~ a contravariant (R -) functor
 Then F is a duality, if there exists
 a contravariant (R -) functor
 $H: \mathcal{D} \rightarrow \mathcal{C}$ such that

$HF \cong \text{id}_{\mathcal{C}}$ and $FH \cong \text{id}_{\mathcal{D}}$
 as (R -) functors.

k field, V vector space

$$D = \text{Hom}_k(-, k): \text{Vec}(k) \rightarrow \text{Vec}(k)$$

is a contravariant functor.

Assume that $\dim_k V = t < \infty$. If $\{v_i\}_{i=1}^t$

is a k -basis for V , then $\{v_i^*\}_{i=1}^t$,
 where $v_i^* \in D(V) = \text{Hom}_k(V, k)$ is given

by $v_i^* \left(\sum_{j=1}^t a_j v_j \right) = a_i, \quad a_j \in k,$

is a basis for $D(V)$ (the dual basis)

$$\Rightarrow \dim_k V = \dim_k D(V) = \dim_k DD(V) \quad (82)$$

Define $\varphi_V: V \rightarrow DD(V) = \text{Hom}_k(\text{Hom}_k(V, k), k)$
 by $v \mapsto \varphi_V(v): \text{Hom}_k(V, k) \rightarrow k$
 where $\varphi_V(v)(f) = f(v)$ for $f \in D(V)$

Exercise: (a) φ_V is \uparrow - \downarrow .

($\Rightarrow \varphi_V$ is an isomorphism for $\dim_k V < \infty$)

(b) $\varphi = \{ \varphi_V \}_{V \in \text{vec}(k)}: \text{id}_{\text{vec}(k)} \rightarrow DD$
 is an isomorphism of functors.

Check: (a) \uparrow fun. $\dim k$ -alg

For $X \in \text{mod } \Lambda$, then

$D(X) = \text{Hom}_k(X, k)$ is a left Λ^{op} module

via $f \leftarrow \lambda \in \Lambda^{\text{op}}, x \in X$

$$(\lambda \cdot f)(x) = f(\lambda x)$$

for $f \in D(X), \lambda \in \Lambda^{\text{op}}$ and $x \in X$.

(b) $f: X \rightarrow Y \in \text{mod } \Lambda$. Then

$$D(f): D(Y) = \text{Hom}_k(Y, k) \rightarrow \text{Hom}_k(X, k) = D(X)$$

$$a \mapsto a \cdot f$$

ρ is a $\Lambda^{\mathcal{P}}$ -homomorphism.

$$\Rightarrow D = \text{Hom}_k(-, k) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{\mathcal{P}}$$

$$\text{and } D = \text{Hom}_k(-, k) : \text{mod } \Lambda^{\mathcal{P}} \rightarrow \text{mod } \Lambda$$

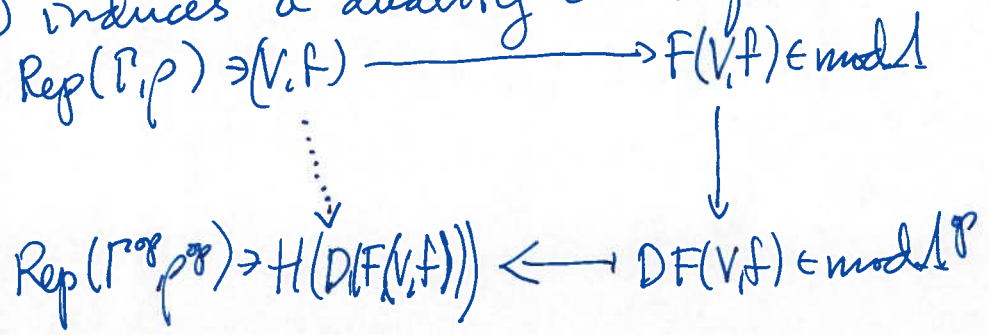
[Note: X fin. gen. Λ -module $\Rightarrow \dim_k X < \infty$
 $\Rightarrow \dim_k D(X) < \infty$
 $\Rightarrow D(X)$ fin. gen. $\Lambda^{\mathcal{P}}$ -module]

Proposition 51

Λ fin. dim k -alg, k field.

Then $D = \text{Hom}_k(-, k) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{\mathcal{P}}$
 is a duality. \square

$\Lambda = k\Gamma / \langle \rho \rangle$ - fin. dim k -alg, k field
 D induces a duality on representations?



(Γ, ρ) , $\exists^t \rho \in \mathcal{J}^t$, $\Gamma_0 = \{1, 2, \dots, n\}$ (83)
 Define $(\Gamma^{\mathcal{P}}, \rho^{\mathcal{P}})$ by

$$\Gamma_0^{\mathcal{P}} = \Gamma_0$$

$\Gamma_1^{\mathcal{P}}$: for each arrow $\alpha: i \rightarrow j$ in Γ_1 ,
 there is an arrow $\alpha^{\mathcal{P}}: j \rightarrow i$ in $\Gamma_1^{\mathcal{P}}$.

If $\rho = \alpha_r \alpha_{r-1} \dots \alpha_2 \alpha_1$ is a path Γ , let
 $\rho^{\mathcal{P}} = \alpha_1^{\mathcal{P}} \alpha_2^{\mathcal{P}} \dots \alpha_{r-1}^{\mathcal{P}} \alpha_r^{\mathcal{P}}$
 in $\Gamma_0^{\mathcal{P}}$. Then $\Lambda^{\mathcal{P}} \cong k\Gamma^{\mathcal{P}} / \langle \rho^{\mathcal{P}} \rangle$ and
 $\text{Rep}(\Gamma^{\mathcal{P}}, \rho^{\mathcal{P}})$ is equivalent to $\text{mod } \Lambda^{\mathcal{P}}$.

Let $(V, f) \in \text{Rep}(\Gamma, \rho)$. Then $F(V, f) \in \text{mod } \Lambda$
 and $DF(V, f) \in \text{mod } \Lambda^{\mathcal{P}}$. Underlying vector
 space of $DF(V, f)$ is

$$D\left(\bigoplus_{i=1}^n V(i)\right).$$

$$HD(F(V, f)) = (V', f') :$$

$$V'(i) = e_i^{\mathcal{P}} D(F(V, f)) = e_i^{\mathcal{P}} \text{Hom}_k\left(\bigoplus_{j=1}^n V(j), k\right)$$

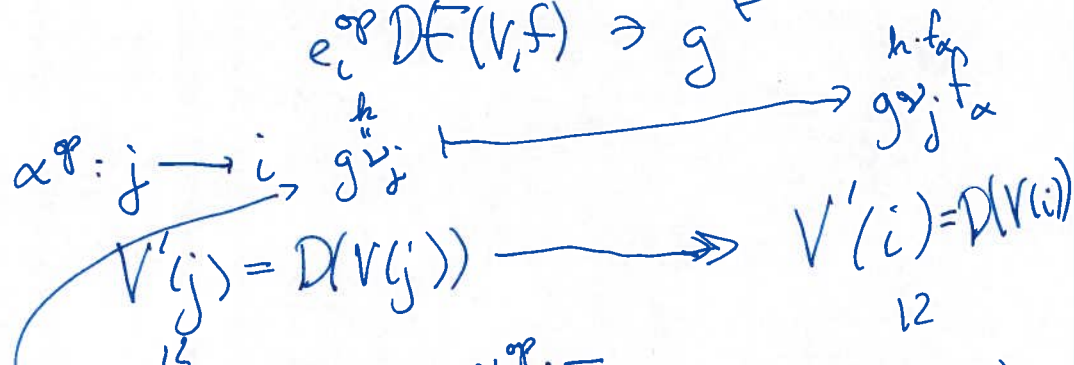
$$v_j : V_j \hookrightarrow \bigoplus_{l=1}^n V(l), \quad g \in \text{Hom}_k\left(\bigoplus_{i=1}^n V(i), k\right)$$

Then

$$\begin{aligned}
 (e_i \otimes g)(v_1, v_2, \dots, v_n) &= g(e_i(v_1, v_2, \dots, v_n)) \\
 &= g(0, \dots, 0, v_i, 0, \dots, 0) \\
 &= g v_i(v_i)
 \end{aligned}$$

$$e_i \otimes g \rightsquigarrow g v_i \in DV(i) = \text{Hom}_k(V(i), k)$$

$$V'(i) = D(V(i)) = \text{Hom}_k(V(i), k) \xrightarrow{g v_i}$$



$$\begin{aligned}
 (\alpha^{\text{op}} g)(v) &= g(\alpha \cdot v) = g(\alpha(v_1, v_2, \dots, v_n)) \\
 &= g(0, \dots, 0, f_{\alpha}(v_i), 0, \dots, 0) \\
 &\quad \uparrow \text{j-th coord.} \\
 &= g v_j(f_{\alpha}(v_i))
 \end{aligned}$$

$$\begin{aligned}
 f'_{\alpha^{\text{op}}} : V'(j) &\xrightarrow{=} D(Vg) \longrightarrow D(V(i)) = V'(i) \\
 &\stackrel{\parallel}{=} D(f'_{\alpha})
 \end{aligned} \tag{84}$$

$$\Rightarrow (V', f') = \left(\{D(V(i))\}_{i=1}^n, \{D(f_{\alpha})\}_{\alpha \in P_i}\right).$$

Exercise: V, W fin. dim vector spaces
 B, B' basis for V and V' , respectively
 B^*, B'^* dual basis for $D(V)$ and $D(W)$
 let $f: V \rightarrow W$. Suppose that

$$m_{B'}^{B'}(f) = A$$

Then for $D(f): D(W) \rightarrow D(V)$
 we have the matrix representation

$$m_{B'^*}^{B^*}(D(f)) = A^T$$

for the dual map $D(f)$.

Example

$\rho: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \rho = \{\gamma(\beta)\}, \Lambda = k[\Gamma/\langle \rho \rangle], k \text{ field}$

$\rho^{\text{op}}: 1 \xleftarrow{\alpha^{\text{op}}} 2 \xleftarrow{\beta^{\text{op}}} 3, \rho^{\text{op}} = \{\beta^{\text{op}}\gamma^{\text{op}}\}$

$\Lambda \bar{e}_1: k \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k \quad D(\Lambda \bar{e}_1): k \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k^2 \xleftarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k$

$\Lambda \bar{e}_2: 0 \xrightarrow{0} k \xrightarrow{1} k \quad D(\Lambda \bar{e}_2): 0 \xleftarrow{0} k \xleftarrow{1} k$

$\Lambda \bar{e}_3: 0 \xrightarrow{0} 0 \xrightarrow{0} k \quad D(\Lambda \bar{e}_3): 0 \xleftarrow{0} 0 \xleftarrow{0} k$

Lemma 52 Λ fin. dim k -alg, k field

(a) $\eta: 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ exact in mod Λ

$\Leftrightarrow 0 \rightarrow D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \rightarrow 0$ exact in mod Λ^{op} .

(b) S simple Λ -module $\Leftrightarrow D(S)$ simple Λ^{op} -module

(c) $l(A) = l(D(A))$ for $A \in \text{mod } \Lambda$.

Sketch of proof: (a) Use that η splits as a sequence of k -modules, that D preserves k -dimension and that

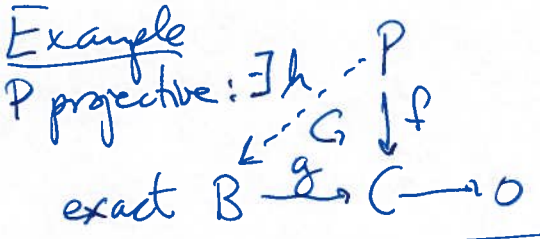
$D^2 \approx \text{id}_{\text{mod } \Lambda}$

(b) Use (a).

(c) Induction on length. □

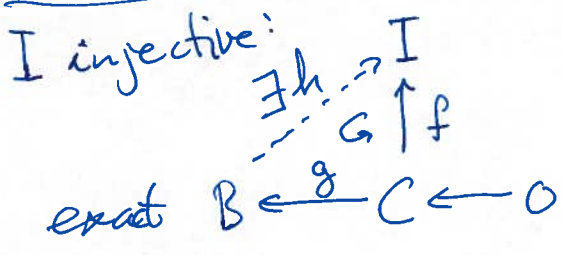
Given a statement S in a category \mathcal{C} , then the dual statement S^* is the statement about \mathcal{C} reversing the direction of all morphisms and replacing all compositions $\alpha\beta$ of morphisms by $\beta\alpha$.

Example



$\forall g, \forall f \Rightarrow \left\{ \begin{array}{l} \exists h: P \rightarrow B \\ \text{such that} \\ gh = f \end{array} \right.$

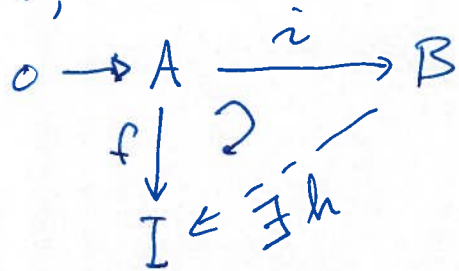
I injective:



$\forall g, \forall f \Rightarrow \left\{ \begin{array}{l} \exists h: B \rightarrow I \\ \text{such that} \\ hg = f \end{array} \right.$

Injective modules

DEF: Λ ring, $I \in \text{Mod } \Lambda$. Then I is injective if for every monomorphism $i: A \rightarrow B$ in $\text{Mod } \Lambda$ and every homomorphism $f: A \rightarrow I$, there exists $h: B \rightarrow I$ such that $hi = f$, i.e.



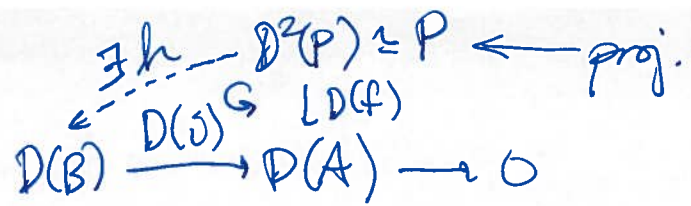
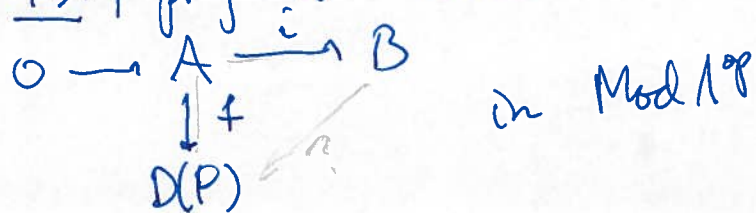
Proposition 53 Λ fin. dim k -alg, k -field.

$P \in \text{mod } \Lambda$

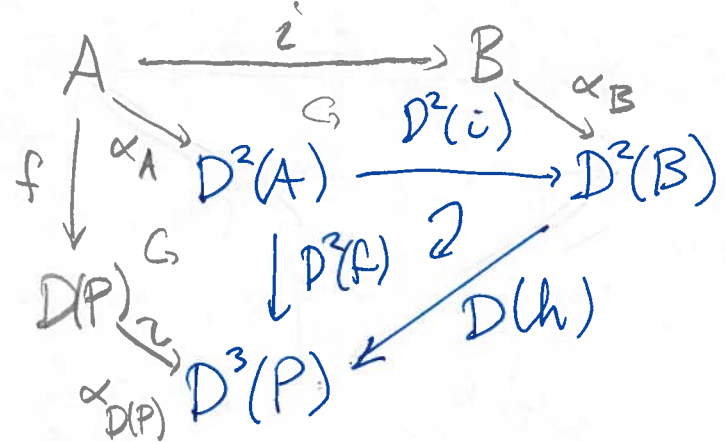
(a) P projective Λ -module $\Leftrightarrow D(P)$ injective Λ^{op} -module

(b) Any Λ -module $M \in \text{mod } \Lambda$ is a submodule of an injective Λ -module in $\text{mod } \Lambda$.

Proof: (a) \Rightarrow : P projective. Consider



Apply D



$$\Rightarrow \alpha_{D(P)} f = D(h) \alpha_B i$$

$$\alpha_{D(P)}^{-1} \cdot \mid f = \alpha_{D(P)}^{-1} \alpha_{D(P)} f = (\alpha_{D(P)}^{-1} D(h) \alpha_B) i$$

$\Rightarrow D(P) \in \text{mod } \Lambda^{\text{op}}$ is injective.

\Leftarrow : Baer Criterion & Λ noetherian \Rightarrow Can restrict to fin. gen. modules

Use dual arguments.

(b) Let $M \in \text{mod } \Lambda$. Then $D(M) \in \text{mod } \Lambda^{\text{op}}$. Let $P \rightarrow D(M)$ be the proj. cover of $D(M)$

Then $M \xrightarrow{\alpha_M} D^2(M) \xrightarrow{\quad} D(P) \in \text{mod } \Lambda$
 \uparrow injective Λ -module \square

Remark: Λ ring, $M \in \text{Mod } \Lambda$

Can be shown: $M \hookrightarrow I$, I injective Λ -module.

However; Even if $M \subseteq A$ a fin. gen Λ -module
 I need not to be fin. gen. Λ -module

(\mathbb{Z} -modules = \mathbb{A}_b : $\mathbb{Z} \hookrightarrow \mathbb{Q}$)

DEF: Λ ring

(a) $A \subseteq X$ Λ -modules. Then A is an essential submodule of X , if for each non-zero submodule B of X , then $A \cap B \neq (0)$.

(b) A monomorphism $i: A \rightarrow X$ is essential if $i(A)$ is an essential submodule of X .

(c) A monomorphism $i: A \rightarrow I$ is an injective envelope if

(i) I is injective.

(ii) i is an essential monomorphism.

Can be shown: (MA3204) Λ ring (87)

Every Λ -module has an injective envelope

DEF: Λ artin R -algebra, $A \in \text{mod } \Lambda$. The socle of A , $\text{soc } A$, is the sum of all simple submodules of A .

Example

$\Gamma: \begin{matrix} & & 1 & & \\ & \swarrow & & \searrow & \\ 2 & & & & 3 \end{matrix}$, k field, $\Lambda = k\Gamma$

$\Lambda e_1 \cong \begin{matrix} & k & \\ & \swarrow & \searrow \\ k & & k \end{matrix} \cong \begin{matrix} 0 & & 0 \\ \swarrow & & \searrow \\ k & 0 & 0 & k \end{matrix} = \Lambda e_2 \oplus \Lambda e_3$
" " " " " " $\text{soc } \Lambda e_1$

Note: (1) In general, $\text{soc } S = S$, when S is a (semi)simple Λ -module.

(2) $\text{soc } A \subseteq A$ is $\left\{ \begin{array}{l} \text{a submodule of } A \\ \text{a semisimple submodule of } A \end{array} \right.$

(3) Λ artin- R -alg, $A \in \text{mod } \Lambda$, $A \neq (0)$
Since $l(A) < \infty$, then \exists a simple Λ -submodule $S \subseteq A$, that is, $\text{soc } A \neq (0)$. Furthermore, $A \neq (0) \Leftrightarrow \text{soc } A \neq (0)$, and $D(A/rA) \cong \text{soc } DA$.

Assume that $\ker \varphi \neq (0)$.

$$\Rightarrow A \cap \ker \varphi \neq (0)$$

$$\Rightarrow \exists a \in A \setminus \{0\} \text{ s.t. } 0 = \varphi(a) = \varphi v_1(a) = v_2(a)$$

$$\downarrow$$

$$a=0 \quad \text{X}$$

$$\Rightarrow \ker \varphi = (0) \Rightarrow I_1 \cong \text{Im } \varphi \subseteq I_2$$

↑ injective.

Recall: $0 \rightarrow I \hookrightarrow B \rightarrow C \rightarrow 0$ exact
& I injective
 $\Rightarrow B = I \oplus B'$

$$\Rightarrow I_2 = \text{Im } \varphi \oplus I_2'$$

$$\Rightarrow A \xrightarrow{v_2 = \varphi v_1} I_2 = \text{Im } \varphi \oplus I_2'$$

and $\text{Im } v_2 \subseteq \text{Im } \varphi v_1$.

$$\Rightarrow A \cap I_2' = (0)$$

A essential submodule of I_2

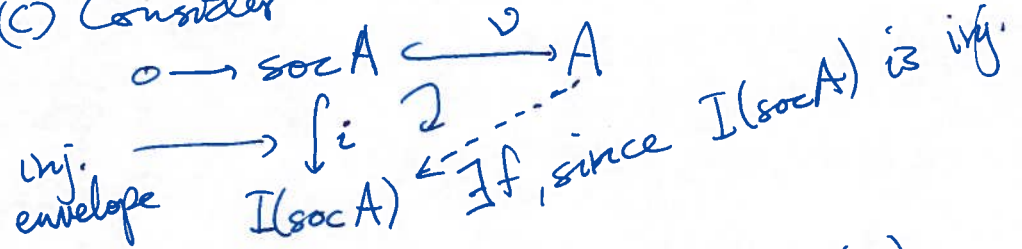
$$\Rightarrow I_2' = (0) \text{ and } \text{Im } \varphi = I_2$$

$\Rightarrow \varphi$ is an isom and therefore inj.

envelopes are ~~isomorphic~~ unique up to (89)

isomorphism,

(c) Consider



• $f v = i \downarrow -1 \Rightarrow \ker f \cap \text{soc } A = (0)$

$\text{soc } A \hookrightarrow A$ ess. $\Rightarrow \ker f = (0) \Rightarrow f \downarrow -1$.

• Let $(0) \neq A' \subseteq I(\text{soc } A)$. WTS: $f(A) \cap A' \neq (0)$.

Have: $f(A) \cap A' \supseteq \text{soc } A \cap A' \neq (0)$, since $\text{soc } A$ is an ess. submodule of $I(\text{soc } A)$.

$$\Rightarrow f(A) \text{ ess. submodule of } I(\text{soc } A)$$

$$\Rightarrow A \xrightarrow{f} I(\text{soc } A) \text{ inj. envelope.}$$

$\text{rad } 1 = \underline{r}$ □

Lemma 56 A artin R -algebra, $A \in \text{mod } R$

$$\text{soc } A = \{a \in A \mid \underline{r} \cdot a = (0)\} = S_A$$

Proof: $\text{soc } A$ semisimple $\Rightarrow \underline{r} \cdot \text{soc } A = (0)$

$$\Rightarrow \text{soc } A \subseteq S_A$$

S_A is a submodule of A .

$E \cdot S_A = (0) \Rightarrow S_A$ is a semisimple submodule of A
 $\Rightarrow S_A \subseteq \text{soc } A \Rightarrow \text{soc } A = S_A$.

Exercise: Λ artin k -algebra, $A_1, A_2 \in \text{mod } \Lambda$

(a) $\text{soc } A \cong \text{Hom}_\Lambda(\Lambda/\mathfrak{r}, A)$

(b) $\text{soc}(A_1 \oplus A_2) = \text{soc } A_1 \oplus \text{soc } A_2$

Proposition 57 Λ fin. dim k -alg, k -field.

$P \xrightarrow{f} A$ is a proj. cover in $\text{mod } \Lambda \Leftrightarrow \begin{cases} D(A) \xrightarrow{D(f)} D(P) \text{ is an} \\ \text{injective envelope} \\ \text{in } \text{mod } \Lambda^{\text{op}}. \end{cases}$

"Proof": Use $X \in \text{mod } \Lambda \Rightarrow \text{soc } D(X) \cong D(X/\mathfrak{r}X)$

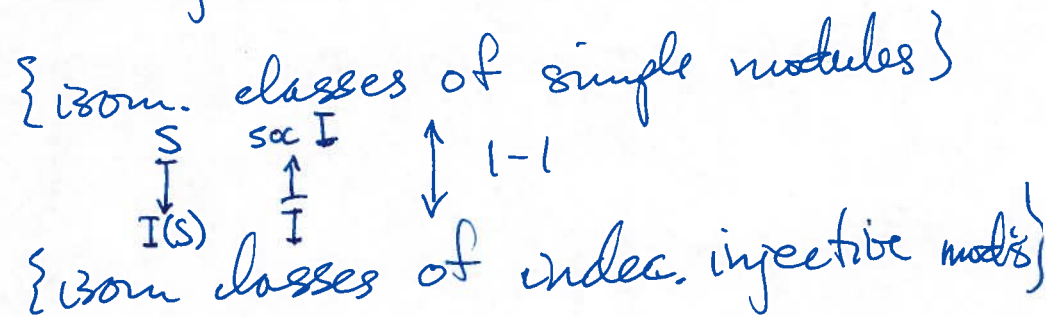
Using duality one can show for Λ a fin. dim k -alg, k -field.

(a) $A, B \in \text{mod } \Lambda \Rightarrow I(A \oplus B) \cong I(A) \oplus I(B)$

(b) I injective in $\text{mod } \Lambda$

I indecomposable $\Leftrightarrow \text{soc } I$ simple (90)

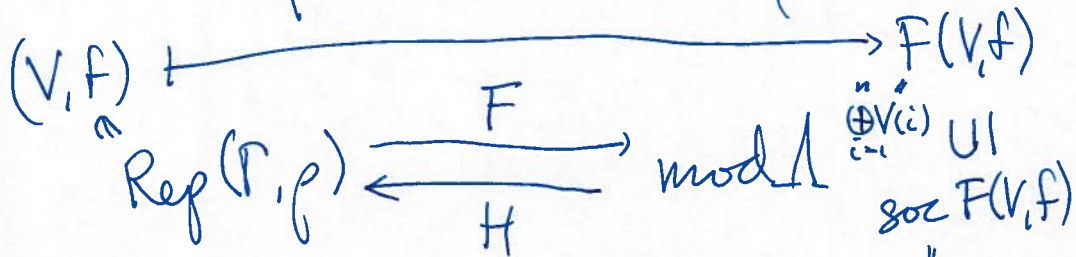
(c) There is a 1-1 correspondence between isomorphism classes of simple Λ -modules and isomorphism classes of indecomposable injective Λ -modules:



The socle of a representation

(Γ, ρ) quiver with relations ρ, k field
 $J^t \in \langle \rho \rangle \in J^2, \Gamma_0 = \{1, 2, \dots, n\}$.

$\Lambda = k\Gamma / \langle \rho \rangle, \Sigma = \text{rad} \Lambda = J / \langle \rho \rangle \subseteq \Lambda$



$m = (v_1, v_2, \dots, v_n) \in F(V, F) \quad \{m \in F(V, F) \mid \Sigma \cdot m = (0)\}$

$m \in \text{soc } F(V, F) \Leftrightarrow \bar{\alpha} \cdot m = 0, \forall \alpha: i \rightarrow j \in \Gamma$
 since $\Sigma = J / \langle \rho \rangle, (J = \langle \text{arrows} \rangle)$

$\Leftrightarrow (0, \dots, 0, \underset{\uparrow j\text{th}}{f_\alpha(v_i)}, 0, \dots, 0) = 0, \forall \alpha: i \rightarrow j \in \Gamma$

$\Leftrightarrow f_\alpha(v_i) = 0, \forall \alpha: i \rightarrow j \in \Gamma$

$\Leftrightarrow v_i \in \bigcap_{\alpha: i \rightarrow j \in \Gamma} \text{Ker } f_\alpha, \forall i = 1, 2, \dots, n$

$H(\text{soc } F(V, F)) = ?$

Let $V'(i) = \bigcap_{\alpha: i \rightarrow j} \text{Ker } f_\alpha \subseteq V(i), \quad (91)$
 and let $f'_\alpha = f_\alpha / V'(i) = 0$.

Example

$\Gamma: 1 \xrightarrow{\alpha} 2 \xrightarrow{\sigma} 3, \rho = \{\sigma\beta\}, k$ field
 $\Lambda = k\Gamma / \langle \rho \rangle$

