Duality

DEF: $\mathcal{C}, \mathcal{D}$ (R-) categories, $F : \mathcal{C} \to \mathcal{D}$ a contravariant (R-) functor. Then F is a <u>duality</u>, if there exists a contravariant (R-) functor $H : \mathcal{D} \to \mathcal{C}$ such that

$$HF \simeq id_{\mathcal{C}} \quad \text{and} \quad FH \simeq id_{\mathcal{D}}$$

as (R-) functors.

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k field, V vector space

$$D = \mathrm{Hom}_k(-, k) : \mathrm{Vec}(k) \to \mathrm{Vec}(k)$$

is a contravariant functor.

Assume that $\dim_k V = t < \infty$. If $\{v_i\}_{i=1}^t$ is a k-basis for V, then $\{v_i^*\}_{i=1}^t$, where $v_i^* \in D(V) = \mathrm{Hom}_k(V, k)$ is given by $v_i^*\left(\sum_{j=1}^t a_j v_j\right) = a_i$, $a_j \in k$, is a basis for $D(V)$ (the dual basis)

$\Rightarrow \dim_k V = \dim_k D(V) = \dim_k DD(V)$

Define $\varphi_V : V \longrightarrow DD(V) = \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$ by

$$v \longmapsto \varphi_V(v) : \underset{D(V)}{\underset{\shortparallel}{\mathrm{Hom}_k(V, k)}} \to k$$

where $\varphi_V(v)(f) = f(v)$ for $f \in D(V)$

Exercise: (a) φ_V is 1-1.

($\Rightarrow \varphi_V$ is an isomorphism for $\dim_k V < \infty$)

(b) $\varphi = \{\varphi_V\}_{V \in \mathrm{vec}(k)} : id_{\mathrm{vec}(k)} \longrightarrow DD$ is an <u>isomorphism of functors.</u>

Check: (a) Λ fin. dim k-alg. For $X \in \mathrm{mod}\,\Lambda$, then $D(X) = \mathrm{Hom}_k(X, k)$ is a left Λ^{op}-module via

$$(\lambda \cdot f)(x) = f(\lambda x)$$

for $f \in D(X)$, $\lambda \in \Lambda^{op}$ and $x \in X$.

(b) $f : X \longrightarrow Y \in \mathrm{mod}\,\Lambda$. Then

$$D(f) : D(Y) = \mathrm{Hom}_k(Y, k) \to \mathrm{Hom}_k(X, k) = D(X)$$

$$g \longmapsto g \cdot f$$

is a Λ^{op}-homomorphism.

$\Rightarrow D = \mathrm{Hom}_k(-,k) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{op}$

and $D = \mathrm{Hom}_k(-,k) : \mathrm{mod}\,\Lambda^{op} \to \mathrm{mod}\,\Lambda$

[Note: X fin. gen. Λ-module $\Rightarrow \dim_k X < \infty$
$\Rightarrow \dim_k D(X) < \infty$
$\Rightarrow D(X)$ fin. gen. Λ^{op}-module]

Proposition 51

Λ fin. dim k-alg, k field.
Then $D = \mathrm{Hom}_k(-,k) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{op}$ is a duality. $\square$

$\Lambda = k\Gamma/\langle\rho\rangle$ - fin. dim k-alg, k field

D induces a duality on representations?

$$\begin{tikzcd}
\mathrm{Rep}(\Gamma,\rho) \ni (V,f) \arrow[r] \arrow[d, dotted] & F(V,f) \in \mathrm{mod}\,\Lambda \arrow[d] \\
\mathrm{Rep}(\Gamma^{op},\rho^{op}) \ni H(D(F(V,f))) & DF(V,f) \in \mathrm{mod}\,\Lambda^{op} \arrow[l]
\end{tikzcd}$$

(Γ,ρ), $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1,2,\dots,n\}$

Define (Γ^{op},ρ^{op}) by

$\Gamma_0^{op} = \Gamma_0$

Γ_1^{op}: for each arrow $\alpha : i \to j$ in Γ_1, there is an arrow $\alpha^{op} : j \to i$ in Γ_1^{op}.

If $\rho = \alpha_r \alpha_{r-1} \cdots \alpha_2 \alpha_1$ is a path Γ, let
$\rho^{op} = \alpha_1^{op} \alpha_2^{op} \cdots \alpha_{r-1}^{op} \alpha_r^{op}$
in Γ^{op}. Then $\Lambda^{op} \simeq k\Gamma^{op}/\langle\rho^{op}\rangle$ and $\mathrm{Rep}(\Gamma^{op},\rho^{op})$ is equivalent to $\mathrm{mod}\,\Lambda^{op}$.

Let $(V,f) \in \mathrm{Rep}(\Gamma,\rho)$. Then $F(V,f) \in \mathrm{mod}\,\Lambda$ and $DF(V,f) \in \mathrm{mod}\,\Lambda^{op}$. Underlying vector space of $DF(V,f)$ is

$$D\Big(\bigoplus_{i=1}^{n} V(i)\Big).$$

$HD(F(V,f)) = (V',f')$:

$$V'(i) = e_i^{op} D(F(V,f)) = e_i^{op} \mathrm{Hom}_k\Big(\bigoplus_{j=1}^{n} V(j), k\Big)$$

$\nu_j : V_j \hookrightarrow \bigoplus_{l=1}^{n} V(l)$, $g \in \mathrm{Hom}_k\Big(\bigoplus_{j=1}^{n} V(j), k\Big)$

Then

$$(e_i^{op} g)(v_1, v_2, \dots, v_n) = g(e_i(v_1, v_2, \dots, v_n))$$
$$= g((0, \dots, 0, v_i, 0, \dots, 0))$$
$$= g\nu_i(v_i)$$

$$e_i^{op} g \longmapsto g\nu_i \in DV(i) = \mathrm{Hom}_k(V(i), k)$$

$$V'(i) = D(V(i)) = \mathrm{Hom}_k(V(i), k) \ni g\nu_i$$

$$e_i^{op} DF(V,f) \ni g$$

$$\alpha^{op} : j \longrightarrow i, \quad \overset{h}{g\nu_j} \longmapsto \overset{h \cdot f_\alpha}{g\nu_j f_\alpha}$$

$$\begin{array}{ccc} V'(j) = D(V(j)) & \longrightarrow & V'(i) = D(V(i)) \\ \wr\wr & & \wr\wr \\ e_j^{op} DF(V,f) & \xrightarrow{\alpha^{op} \cdot -} & e_i^{op} DF(V,f) \\ g & \longmapsto & g\nu_j(f_\alpha(v_i)) \end{array}$$

$$(\alpha^{op} g)(v) = g(\alpha \cdot v) = g(\alpha(v_1, v_2, \dots, v_n))$$
$$= g((0, \dots, 0, \underset{\uparrow\ j\text{-th coord.}}{f_\alpha(v_i)}, 0, \dots, 0))$$
$$= g\nu_j(f_\alpha(v_i))$$

$$f'_{\alpha^{op}} : V'(j) = D(V(j)) \longrightarrow D(V(i)) = V'(i)$$
$$f'_{\alpha^{op}} = D(f_\alpha)$$

$$\Longrightarrow (V', f') = \left(\{D(V(i))\}_{i=1}^{n}, \{D(f_\alpha)\}_{\alpha \in \Gamma_1} \right).$$

<u>Exercise</u>: V, W fin. dim vector spaces
B, B' basis for V and V', respectively
B^*, B'^* dual basis for $D(V)$ and $D(W)$
Let $f : V \longrightarrow W$. Suppose that

$$m_B^{B'}(f) = A$$

Then for $D(f) : D(W) \longrightarrow D(V)$ we have the matrix representation

$$m_{B'^*}^{B^*}(D(f)) = A^T$$

for the dual map $D(f)$.

Example

$\Gamma: 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$, $\rho = \{\gamma\beta\}$, $\Lambda = k\Gamma/\langle\rho\rangle$, k field

$\Gamma^{op}: 1 \overset{\alpha^{op}}{\underset{\beta^{op}}{\leftleftarrows}} 2 \xleftarrow{\gamma^{op}} 3$, $\rho^{op} = \{\beta^{op}\gamma^{op}\}$

$$\Lambda\bar{e}_1: k \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\rightrightarrows}} k^2 \xrightarrow{(1\ 0)} k \qquad D(\Lambda\bar{e}_1): k \overset{(1\ 0)}{\underset{(0\ 1)}{\leftleftarrows}} k^2 \xleftarrow{\binom{1}{0}} k$$

$$\Lambda\bar{e}_2: 0 \overset{0}{\underset{}{\rightrightarrows}} k \xrightarrow{1} k \qquad D(\Lambda\bar{e}_2): 0 \overset{0}{\underset{0}{\leftleftarrows}} k \xleftarrow{1} k$$

$$\Lambda\bar{e}_3: 0 \overset{0}{\underset{0}{\rightrightarrows}} 0 \xrightarrow{0} k \qquad D(\Lambda\bar{e}_3): 0 \overset{0}{\underset{0}{\leftleftarrows}} 0 \xleftarrow{0} k$$

Lemma 52 Λ fin. dim k-alg, k field

(a) $\eta: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact in $\mathrm{mod}\,\Lambda$

$\iff 0 \to D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \to 0$ exact in $\mathrm{mod}\,\Lambda^{op}$.

(b) S simple Λ-module $\iff D(S)$ simple Λ^{op}-module

(c) $l(A) = l(D(A))$ for $A \in \mathrm{mod}\,\Lambda$.

Sketch of proof: (a) Use that η splits as a sequence of k-modules, that D preserves k-dimension and that $D^2 \simeq \mathrm{id}_{\mathrm{mod}\,\Lambda}$

(b) Use (a).

(c) Induction on length. $\square$

Given a statement S in a category $\mathcal{C}$, then <u>the dual statement S^*</u> is the statement about $\mathcal{C}$ reversing the direction of all morphisms and replacing all compositions $\alpha\beta$ of morphisms by $\beta\alpha$.

Example

P projective:
$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$
exact

$\forall g, \forall f \Rightarrow \begin{cases} \exists h: P \to B \\ \text{such that} \\ gh = f \end{cases}$

I injective:
$$\begin{tikzcd} & I & \\ B \arrow[ur, dashed, "\exists h"] & C \arrow[l, "g"'] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd}$$
exact

$\forall g, \forall f \Rightarrow \begin{cases} \exists h: B \to I \\ \text{such that} \\ hg = f \end{cases}$

Injective modules.

DEF: Λ ring, $I \in \operatorname{Mod}\Lambda$. Then I is *injective* if for every monomorphism $i: A \to B$ in $\operatorname{Mod}\Lambda$ and every homomorphism $f: A \to I$, there exists $h: B \to I$ such that $hi = f$, i.e.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[dl, dashed, "\exists h"] \\ & I & \end{tikzcd}$$

Proposition 53 Λ fin. dim k-alg, k-field. $P \in \operatorname{mod}\Lambda$

(a) P projective Λ-module $\Longleftrightarrow$ $D(P)$ injective Λ^{op}-module

(b) Any Λ-module $M \in \operatorname{mod}\Lambda$ is a submodule of an injective Λ-module in $\operatorname{mod}\Lambda$.

Proof: (a) $\Longrightarrow$: P projective. Consider

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[dl, dashed, "?"] \\ & D(P) & \end{tikzcd}$$

in $\operatorname{Mod}\Lambda^{op}$

Apply D

$$\begin{tikzcd} & D^2(P) \cong P \arrow[d, "D(f)"] \arrow[dl, dashed, "\exists h"'] & \arrow[l] \text{proj.} \\ D(B) \arrow[r, "D(i)"] & D(A) \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "\alpha_A"] \arrow[dd, "f"'] & & B \arrow[dr, "\alpha_B"] & \\ & D^2(A) \arrow[rr, "D^2(i)"] \arrow[dd, "D^2(f)"] & & D^2(B) \arrow[dll, "D(h)"] \\ D(P) \arrow[dr, "\alpha_{D(P)}", "\sim"'] & & & \\ & D^3(P) & & \end{tikzcd}$$

$\Longrightarrow \alpha_{D(P)} f = D(h)\alpha_B i$

$\alpha_{D(P)}^{-1} \cdot \Big|\quad f = \alpha_{D(P)}^{-1}\alpha_{D(P)} f = \left(\alpha_{D(P)}^{-1} D(h)\alpha_B\right) i$

$\Longrightarrow D(P) \in \operatorname{mod}\Lambda^{op}$ is injective.

$\Longleftarrow$: Baer Criterion & Λ noetherian
$\Longrightarrow$ Can restrict to fin. gen. modules
Use dual arguments.

(b) Let $M \in \operatorname{mod}\Lambda$. Then $D(M) \in \operatorname{mod}\Lambda^{op}$. Let $P \twoheadrightarrow D(M)$ be the proj. cover of $D(M)$. Then

$M \overset{\alpha_M}{\cong} D^2(M) \rightarrowtail D(P) \in \operatorname{mod}\Lambda$
injective Λ-module $\square$

Remark: Λ ring, $M \in \operatorname{Mod}\Lambda$

Can be shown: $M \hookrightarrow I$, I injective Λ-module.

However; Even if M is a fin-gen Λ-module I need not to be fin.gen. Λ-module

($\mathbb{Z}$-modules $=$ Ab : $\mathbb{Z} \hookrightarrow \mathbb{Q}$)

DEF: Λ ring

(a) $A \subseteq X$ Λ-modules. Then A is an <u>essential submodule of X</u>, if for each non-zero submodule B of X, then $A \cap B \neq (0)$.

(b) A monomorphism $i: A \to X$ is essential if $i(A)$ is an essential submodule of X.

(c) A monomorphism $i: A \to I$ is an <u>injective envelope</u> if

(i) I is injective,

(ii) i is an essential monomorphism.

Can be shown: (MA3204) Λ ring

Every Λ-module has an injective envelope.

DEF: Λ artin R-algebra, $A \in \operatorname{mod}\Lambda$. The <u>socle of A</u>, $\operatorname{soc} A$, is the sum of all simple submodules of A.

Example

Γ: $2 \leftarrow 1 \rightarrow 3$, k field, $\Lambda = k\Gamma$

$$\Lambda e_1 \rightsquigarrow \begin{array}{ccc} & k & \\ {}^{1}\swarrow & & \searrow^{1} \\ k & & k \end{array} \hookleftarrow \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ k & & 0 \end{array} + \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 0 & & k \end{array} = \underbrace{\Lambda e_2 \oplus \Lambda e_3}_{=\ \operatorname{soc}\Lambda e_1}$$

Note: (1) In general, $\operatorname{soc} S = S$, when S is a (semi)simple Λ-module.

(2) $\operatorname{soc} A \subseteq A$ is $\begin{cases} \text{a submodule of } A \\ \text{a semisimple submodule of } A \end{cases}$

(3) Λ artin R-alg, $A \in \operatorname{mod}\Lambda$, $A \neq (0)$

Since $l(A) < \infty$, then $\exists$ a simple Λ-submodule $S \subseteq A$, that is, $\operatorname{soc} A \neq (0)$. Furthermore, $A \neq (0) \Leftrightarrow \operatorname{soc} A \neq (0)$, and $D(A/\mathfrak{r}A) \simeq \operatorname{soc} DA$).

Lemma 54 Λ artin R-algebra, $A \subseteq X \in \operatorname{mod}\Lambda$

TFAE:

(a) A essential submodule of X

(b) $\operatorname{soc} X \subseteq A$

(c) $\operatorname{soc} A = \operatorname{soc} X$.

Proof: (a) $\Rightarrow$ (c): $A \subseteq X$ essential submodule

$S \subseteq X$ simple $\Rightarrow S \neq (0)$

$$\Rightarrow (0) \neq S \cap A \subseteq S$$

S simple $\rightarrow S \cap A = S \subseteq A$

$$\Rightarrow \sum_{\substack{S \subseteq X \\ S \text{ simple}}} S = \operatorname{soc} X \subseteq A$$

$A \subseteq X \Rightarrow \operatorname{soc} A \subseteq \operatorname{soc} X$

Hence, $\operatorname{soc} A = \operatorname{soc} X$.

(c) $\Rightarrow$ (b): Obviously, $\operatorname{soc} A = \operatorname{soc} X$

$$\Rightarrow \operatorname{soc} X \subseteq \operatorname{soc} A \subseteq A.$$

(b) $\Rightarrow$ (a): Assume that $\operatorname{soc} X \subseteq A$.

Let $(0) \neq B \subseteq X$. Then $(0) \neq \operatorname{soc} B \subseteq \operatorname{soc} X$

Hence, $(0) \neq \operatorname{soc} B = \operatorname{soc} B \cap \operatorname{soc} X \subseteq \operatorname{soc} B \cap A \subseteq B \cap A \Rightarrow B \cap A \neq (0)$

$\Rightarrow A$ essential submodule of X. $\square$

Proposition 55 Λ artin R-algebra, $(0) \neq A \in \operatorname{mod}\Lambda$

(a) $A \hookrightarrow I$ injective envelope

$$\Longleftrightarrow \begin{cases} A \hookrightarrow I, & I \text{ injective} \\ \operatorname{soc} A = \operatorname{soc} I \end{cases}$$

(b) Injective envelopes are unique up to isomorphism.

(c) Injective envelope of A

$\simeq$ Injective envelope of $\operatorname{soc} A$.

Proof: (a) Use the definition of an injective envelope and Lemma 54.

(b) Let $A \xhookrightarrow{\nu_i} I_i$ be two injective envelopes

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r, hook, "\nu_1"] \arrow[dr, "\nu_2"'] & I_1 \arrow[d, dashed, "\exists\varphi"] \\ & & I_2 \end{tikzcd}$$

Assume that $\ker\varphi \neq (0)$.

$\Rightarrow A \cap \ker\varphi \neq (0)$

$\Rightarrow \exists a \in A \setminus \{0\}$ s.th. $0 = \varphi(a) = \varphi\nu_1(a)$
$= \nu_2(a)$
$\Downarrow$
$a = 0$ ↯.

$\Rightarrow \ker\varphi = (0) \Rightarrow I_1 \simeq \operatorname{Im}\varphi \subseteq I_2$
injective.

<u>Recall</u>: $0 \to I \hookrightarrow B \to C \to 0$ exact
& I injective
$\Rightarrow B = I \oplus B'$

$\Rightarrow I_2 = \operatorname{Im}\varphi \oplus I_2'$

$\Rightarrow A \xhookrightarrow{\nu_2 = \varphi\nu_1} I_2 = \operatorname{Im}\varphi \oplus I_2'$
and $\operatorname{Im}\nu_2 \subseteq \operatorname{Im}\varphi\nu_1$.

$\Rightarrow A \cap I_2' = (0)$

A essential submodule of I_2

$\Rightarrow I_2' = (0)$ and $\operatorname{Im}\varphi = I_2$

$\Rightarrow \varphi$ is an isom and therefore inj.

envelopes are ~~isomorphic~~ unique up to isomorphism. (89)

(c) Consider

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{soc} A \arrow[r, hook, "\nu"] \arrow[d, hook, "i"] & A \arrow[dl, dashed, "\exists f"] \\ & I(\operatorname{soc} A) & \end{tikzcd}$$

inj. envelope; $\exists f$, since $I(\operatorname{soc} A)$ is inj.

- $f\nu = i$ 1-1 $\Rightarrow \ker f \cap \operatorname{soc} A = (0)$
$\operatorname{soc} A \hookrightarrow A$ ess. $\Rightarrow \ker f = (0) \Rightarrow f$ 1-1.
- Let $(0) \neq A' \subseteq I(\operatorname{soc} A)$. WTS: $f(A) \cap A' \neq (0)$.

<u>Have</u>: $f(A) \cap A' \supseteq \operatorname{soc} A \cap A' \neq (0)$, since
$\operatorname{soc} A$ is an ess. submodule of $I(\operatorname{soc} A)$.

$\Rightarrow f(A)$ ess. submodule of $I(\operatorname{soc} A)$

$\Rightarrow A \xrightarrow{f} I(\operatorname{soc} A)$ inj. envelope. $\square$

$\operatorname{rad}\Lambda = \underline{r}$

<u>Lemma 56</u> Λ artin R-algebra, $A \in \operatorname{mod}\Lambda$

$\operatorname{soc} A = \{a \in A \mid \underline{r} \cdot a = (0)\}. = S_A$

<u>Proof</u>: $\operatorname{soc} A$ semisimple $\Rightarrow \underline{r} \cdot \operatorname{soc} A = (0)$
$\Rightarrow \operatorname{soc} A \subseteq S_A$

S_A is a submodule of A.

$\underline{r} \cdot S_A = (0) \Rightarrow S_A$ is a semisimple submodule of A.

$\Rightarrow S_A \subseteq \operatorname{soc} A \Rightarrow \operatorname{soc} A = S_A$.

<u>Exercise:</u> Λ artin R-algebra, $A, A_1, A_2 \in \operatorname{mod} \Lambda$

(a) $\operatorname{soc} A \simeq \operatorname{Hom}_\Lambda(\Lambda/\underline{r}, A)$

(b) $\operatorname{soc}(A_1 \oplus A_2) = \operatorname{soc} A_1 \oplus \operatorname{soc} A_2$

<u>Proposition 57</u> Λ fin. dim k-alg, k-field.

$$\left.\begin{array}{l} P \xrightarrow{f} A \text{ is a} \\ \text{proj. cover in} \\ \operatorname{mod} \Lambda \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D(A) \xrightarrow{D(f)} D(P) \text{ is an} \\ \text{injective envelope} \\ \text{in } \operatorname{mod} \Lambda^{op}. \end{array}\right.$$

<u>"Proof":</u> Use $X \in \operatorname{mod} \Lambda \Rightarrow \operatorname{soc} D(X) \simeq D(X/\underline{r}X)$

Using duality one can show for Λ a fin. dim k-alg, k-field.

(a) $A, B \in \operatorname{mod} \Lambda \Rightarrow I(A \oplus B) \simeq I(A) \oplus I(B)$

(b) I injective in $\operatorname{mod} \Lambda$

I indecomposable $\Leftrightarrow \operatorname{soc} I$ simple

(c) There is a 1-1 correspondence between isomorphism classes of simple Λ-modules and isomorphism classes of indecomposable injective Λ-modules:

$$\begin{array}{ccc} \{\text{isom. classes of simple modules}\} & S \qquad \operatorname{soc} I & \\ \updownarrow \; 1\text{-}1 & \downarrow \qquad\quad \uparrow & \\ \{\text{isom classes of indec. injective mod's}\} & I(S) \qquad I & \end{array}$$

The socle of a representation

(Γ,ρ) quiver with relations ρ, k field

$J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0=\{1,2,\dots,n\}$.

$\Lambda = k\Gamma/\langle\rho\rangle$, $\mathfrak{r} = \operatorname{rad}\Lambda = J/\langle\rho\rangle \subseteq \Lambda$

$$(V,f) \longmapsto F(V,f)$$

$$\operatorname{Rep}(\Gamma,\rho) \underset{H}{\overset{F}{\rightleftarrows}} \operatorname{mod}\Lambda \qquad \bigoplus_{i=1}^{n} V(i) \supseteq \operatorname{soc} F(V,f) = \{m\in F(V,f) \mid \mathfrak{r}\cdot m=(0)\}$$

$m=(v_1,v_2,\dots,v_n)\in F(V,f)$

$m\in \operatorname{soc}F(V,f) \iff \bar\alpha\cdot m = 0,\ \forall \alpha: i\to j\in\Gamma_1$

since $\mathfrak{r} = J/\langle\rho\rangle$, ($J$=arrows)

$\iff (0,\dots,0,\underbrace{f_\alpha(v_i)}_{j\text{th}},0,\dots,0)=0,\ \forall\alpha:i\to j\in\Gamma_1$

$\iff f_\alpha(v_i)=0,\ \forall\alpha:i\to j\in\Gamma_1$

$\iff v_i \in \bigcap_{\alpha:i\to j\in\Gamma_1}\operatorname{Ker} f_\alpha,\ \forall i=1,2,\dots,n$

$H(\operatorname{soc}F(V,f))=\,?$

Let $V'(i)=\bigcap_{\alpha:i\to j}\operatorname{Ker} f_\alpha \subseteq V(i)$,

and let $f'_\alpha = f_\alpha|_{V'(i)} = 0$.

Example

$\Gamma: 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$, $\rho=\{\gamma\beta\}$, k field

$\Lambda = k\Gamma/\langle\rho\rangle$

$$\Lambda\bar e_1:\quad \begin{tikzcd} k \arrow[d, shift right, "\binom{1}{0}"'] \arrow[d, shift left, "\binom{0}{1}"] \\ k^2 \arrow[d, "(0\ 1)"] \\ k \end{tikzcd} \qquad \operatorname{soc}\Lambda\bar e_1:\quad \begin{tikzcd} 0 \arrow[d, shift right, "0"'] \arrow[d, shift left, "0"] \\ k\oplus(0) \arrow[d, "0"] \\ k \end{tikzcd}$$