LECTURE NOTES FOR MA3203 RING THEORY

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Contents

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1. Quivers

1.1. Quivers, vertices, arrows and paths.

Definition 1.1. A *quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph,

$$\Gamma_0 = \{\textit{vertices}\}(= \{1, 2, \dots, n\}).$$
$$\Gamma_1 = \{\textit{arrows}\}.$$

We always assume that Γ_0 and Γ_1 are finite sets.

Example 1.2. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, $\Gamma_0 = \{1, 2\}$ and $\Gamma_1 = \{\alpha\}$.

Example 1.3. $\Gamma\colon$ $\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$, $\Gamma_0 = \{1\}$ and $\Gamma_1 = \{\alpha\}$.

Example 1.4. $\Gamma\colon$
$$\begin{tikzcd} 1 \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] & & 2 \arrow[loop right, "\gamma"] \arrow[dl, bend right, "\delta"'] \\ & 3 \arrow[ul, "\theta"] \arrow[ur, bend right, "\epsilon"'] & \end{tikzcd}$$
, $\Gamma_0 = \{1, 2, 3\}$ and $\Gamma_1 = \{\alpha, \beta, \gamma, \delta, \epsilon, \theta\}$.

Have maps: $\mathfrak{s}, \mathfrak{e}\colon \Gamma_1 \to \Gamma_0$

$$\mathfrak{s}(\alpha) = \text{the vertex where } \alpha \in \Gamma_1 \text{ starts},$$
$$\mathfrak{e}(\alpha) = \text{the vertex where } \alpha \in \Gamma_1 \text{ ends}.$$

Definition 1.5. $\Gamma = (\Gamma_0, \Gamma_1)$ quiver. A *path* in Γ is either

(i) an ordered sequence of arrows $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$, where

$$\mathfrak{e}(\alpha_t) = \mathfrak{s}(\alpha_{t+1})$$

for $t = 1, 2, \dots, n-1$ (*non-trivial path*) or

(ii) e_i for each i in Γ_0 (*trivial path*).

In addition,

$$\mathfrak{s}(p) = \mathfrak{s}(\alpha_1) \qquad \mathfrak{s}(e_i) = i$$
$$\mathfrak{e}(p) = \mathfrak{e}(\alpha_n) \qquad \mathfrak{e}(e_i) = i$$

Example 1.6. $\Gamma\colon$
$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\gamma"] & 3 \\ & 4 & \end{tikzcd}$$

Paths:

(i) $\alpha, \beta, \gamma, \beta\alpha, \gamma\alpha$.

(ii) e_1, e_2, e_3, e_4.

Example 1.7. $\Gamma\colon$ $\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$.

Paths:

(i) $\alpha, \alpha^2 = \alpha\alpha, \alpha^3 = \alpha\alpha\alpha, \dots$.

(ii) e_1.

1.2. **Path algebras.** Given $\Gamma = (\Gamma_0, \Gamma_1)$, a quiver, and k a field.

The *path algebra* $k\Gamma$: $k\Gamma$ is the vector space with all the paths in Γ as a basis.

The elements in $k\Gamma$:

$$a_1 p_1 + a_2 p_2 + \cdots + a_t p_t$$

where $a_i \in k$ and p_i are paths in Γ. We write just p for $1p$, when p is a path in Γ.

Example 1.8. Continuing Example 1.6:

$$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5\alpha + a_6\beta + a_7\gamma + a_8\beta\alpha + a_9\gamma\alpha$$
$$y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5\alpha + b_6\beta + b_7\gamma + b_8\beta\alpha + b_9\gamma\alpha$$

$$\begin{aligned} x + y = {}& (a_1 + b_1)e_1 + (a_2 + b_2)e_2 + (a_3 + b_3)e_3 + (a_4 + b_4)e_4 + (a_5 + b_5)\alpha \\ &+ (a_6 + b_6)\beta + (a_7 + b_7)\gamma + (a_8 + b_8)\beta\alpha + (a_9 + b_9)\gamma\alpha \end{aligned}$$

Multiplication. p, q paths in Γ:

(1) p, q both non-trivial

$$p \cdot q = \begin{cases} pq, & \text{if } \mathfrak{e}(q) = \mathfrak{s}(p) \\ 0, & \text{otherwise} \end{cases}$$

(2) p non-trivial, q trivial, $q = e_i$

$$p \cdot q = \begin{cases} p, & \text{if } \mathfrak{s}(p) = i = \mathfrak{e}(q) \\ 0, & \text{otherwise} \end{cases}$$

$$q \cdot p = \begin{cases} p, & \text{if } \mathfrak{e}(p) = i = \mathfrak{s}(q) \\ 0, & \text{otherwise} \end{cases}$$

(3) $p = e_i$, $q = e_j$ (both trivial)

$$p \cdot q = \begin{cases} e_i, & \text{if } \mathfrak{e}(q) = j = i = \mathfrak{s}(p) \\ 0, & \text{otherwise} \end{cases}$$

This is extended distributively to an operation on $k\Gamma$ (see [1, page 50]).

Example 1.9. Γ: $1 \xrightarrow{\alpha} 2$, k field.

Elements in $k\Gamma$: $a_1e_1 + a_2e_2 + a_3\alpha = y$.

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

$$\begin{aligned} (e_1 + e_2) \cdot y &= (e_1 + e_2)(a_1e_1 + a_2e_2 + a_3\alpha) \\ &= a_1e_1^2 + a_2\underbrace{e_1e_2}_{=0} + a_3\underbrace{e_1\alpha}_{=0} + a_1\underbrace{e_2e_1}_{=0} + a_2e_2^2 + a_3e_2\alpha \\ &= a_1e_1 + a_2e_2 + a_3\alpha = y \end{aligned}$$

Similarly $y \cdot (e_1 + e_2) = y$. Hence, $e_1 + e_2$ acts like 1 in $k\Gamma$.

Basis for $k\Gamma$: $\{e_1, e_2, \alpha\}$, $\dim_k k\Gamma = 3$.

Example 1.10. Γ: $1 \circlearrowright \alpha$, and k a field.

$k\Gamma$ has basis: $\{e_1, \alpha, \alpha^2, \alpha^3, \ldots\}$, that is, $\dim_k k\Gamma = \infty$.

Elements in $k\Gamma$: $a_0e_1 + a_1\alpha + a\alpha^2 + \cdots + a_t\alpha^t$, with a_i in k and $t \geqslant 0$.

Note 1.11. (1) In general, $\{e_i\}_{i\in\Gamma}$ are *orthogonal idempotents* in $k\Gamma$, that is,

$$\begin{cases} e_i^2 = e_i \\ e_ie_j = 0 \text{ for } i \neq j \end{cases}$$

(2) Suppose $\Gamma_0 = \{1, 2, \dots, n\}$. Then $e_1 + e_2 + \cdots + e_n$ acts like 1 in $k\Gamma$. Enough to show that

$$p = (e_1 + e_2 + \cdots + e_n)p = p(e_1 + e_2 + \cdots + e_n)$$

for any path p. Suppose that $\mathfrak{s}(p) = i$ and $\mathfrak{e}(p) = j$. Then

$$(e_1 + e_2 + \cdots + e_n)p = e_1p + e_2p + \cdots + e_jp + \cdots + e_np = e_jp \overset{\text{def}}{=} p$$

$$p(e_1 + e_2 + \cdots + e_n) = pe_1 + pe_2 + \cdots + pe_i + \cdots + pe_n = e_jp \overset{\text{def}}{=} p$$

$$\Longrightarrow e_1 + e_2 + \cdots + e_n = 1_{k\Gamma} = \text{identity in } k\Gamma$$

Can show: $k\Gamma$ is a k-algebra with $e_1 + e_2 + \cdots + e_n$ as an identity (see [1, page 50]).

Recall: Λ ring, k field.

Definition 1.12. Λ is a k-*algebra*, if Λ is a vector space over k ($k \times \Lambda \longrightarrow \Lambda$, Λ is a module over k, $\alpha \in k, \lambda \in \Lambda, \alpha \cdot \lambda$) and

$$\alpha(\lambda \cdot \lambda^{'}) = (\alpha \cdot \lambda) \cdot \lambda^{'} = \lambda(\alpha \cdot \lambda^{'})$$

$\forall \alpha \in k, \forall \lambda, \lambda' \in \Lambda$.

Note 1.13. Λ is a k-algebra, if $\exists\ \phi\colon k \to \Lambda$ a ring homomorphism such that

$$\operatorname{Im}\phi \subseteq Z(\Lambda) = \{z \in \Lambda \mid z\lambda = \lambda z, \forall \lambda \in \Lambda\}$$

($\iff \exists R \subseteq \Lambda$ subring such that $R \simeq k$ with $R \subseteq Z(\Lambda)$, just define $\phi(a) = a \cdot 1_\Lambda$).

For $k\Gamma$ the ring homomorphism $\phi\colon k \to k\Gamma$ is given by $\phi(a) = ae_1 + ae_2 + \cdots + ae_n$

Exercise 1.14. (1) $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field.
Find a k-algebra isomorphism

$$\psi\colon k\Gamma \to \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}.$$

(2) $\Gamma\colon\ 1 \circlearrowright \alpha$, k field.
Show that $k\Gamma \simeq k[x]$ as k-algebras.

Definition 1.15. A non-trivial path p in Γ is an *oriented cycle* if

$$\mathfrak{e}(p) = \mathfrak{s}(p).$$

Example 1.16. $\Gamma\colon\ \alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2$

Cycles. $\alpha, \alpha^3, \gamma\beta\alpha, \beta\alpha^{10}\gamma, \dots$ $\dim_k k\Gamma = \infty$

Proposition 1. $\Gamma = (\Gamma_0, \Gamma_1)$ *quiver, k field.*

$$\dim_k k\Gamma < \infty \textit{ if and only } \Gamma \textit{ has no oriented cycles.}$$

Proof. Exercise. □

Proposition 2. *Assume that* $\Gamma = (\Gamma_0, \Gamma_1)$ *has no oriented cycles.*

$$k\Gamma \textit{ is semisimple } \iff \Gamma_1 = \emptyset.$$

Proof. Proposition 1 $\implies \dim_k k\Gamma < \infty \implies k\Gamma$ is a left artinian ring.
$k\Gamma$ semisimple $\iff$ no non-zero nilpotent left ideals in $k\Gamma$.

$\Rightarrow$: Assume that $\Gamma_1 \neq \emptyset$. Let α_1 be an arrow in Γ. Want to find a vertex where at least one arrow ends and no arrow starts. If $\mathfrak{e}(\alpha_1)$ is such a vertex, we are done. If not, there is an arrow α_2 starting in $\mathfrak{e}(\alpha_1)$. If also $\mathfrak{e}(\alpha_2)$ is not as above, we continue. Since Γ has no oriented cycles and Γ is finite, we must end up in a vertex v, where arrows only end and no arrows starts. Say, $\alpha = \alpha_t$ is an arrow ending in v. Then consider $k\Gamma\alpha = k\alpha$. Since

$$(a_1\alpha)(a_2\alpha = (a_1a_2)\underbrace{(\alpha\alpha)}_{=0} = 0 \qquad \implies \qquad (k\Gamma\alpha)^2 = (0)$$

and $k\Gamma\alpha \neq (0)$, we infer that $k\Gamma$ is not semisimple.
$\Leftarrow$: Assume that $\Gamma_1 = \emptyset$. Then

$$\Gamma\colon\ 1 \qquad 2 \cdots\cdots n \quad (n \text{ vertices})$$

Basis for $k\Gamma$: $\{e_1, e_2, \ldots, e_n\}$. Elements in $k\Gamma$: $a_1e_1 + a_2e_2 + \cdots + a_ne_n$ with $a_i \in k$. Have a ring homomorphism

$$\psi\colon \underbrace{k \times \cdots \times k}_{n} \to k\Gamma$$

given by

$$\psi(a_1, a_2, \ldots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n$$

(check this!). Show that ψ is an isomorphism. Therefore $k\Gamma$ is semisimple, since $k\Gamma$ is isomorphic to a finite product of full matrix rings over divisjon rings. $\square$

Note: $k\Gamma$ is not always semisimple, but some factor of $k\Gamma$ is.

Proposition 3. $\Gamma = (\Gamma_0, \Gamma_1)$ *quiver, k field. Let*

$$J = \{\textit{all linear combinations of non-trivial paths}\}.$$

Then J is an ideal in $k\Gamma$ and $k\Gamma/J \simeq \underbrace{k \times \cdots \times k}_{|\Gamma_0|}$, *-semisimple*

Sketch of proof. Define $\psi\colon k\Gamma \to \underbrace{k \times \cdots \times}_{|\Gamma_0| = n} = k^n$

$\psi(a_1e_1 + a_2e_2 + \cdots + a_ne_n +$ linear combinations of non-trivial paths$) = (a_1, a_2, \ldots, a_n)$

<u>Check:</u>

(1) ψ is well-defined,
(2) ψ homomorphism of rings,
(3) $\ker\psi = J$.

$\implies k\Gamma/J \simeq \operatorname{Im}\psi = k^n$. $\square$

2. Modules

Example 2.1. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field.
What is a module over $k\Gamma$?
Let M be a left $k\Gamma$-module.

Recall: $1_{k\Gamma} = e_1 + e_2$,

$$e_ie_j = \begin{cases} e_i^2 = e_i, \\ e_ie_j = 0, \text{ for } i \neq j. \end{cases}$$

Claim: $M = e_1M \oplus e_2M$ as vector space over k.

Proof.

$$\begin{aligned} m &= 1_{k\Gamma} * m \\ &= (e_1 + e_2)m \\ &= e_1m + e_2m \in e_1M + e_2M \\ \implies M &\subseteq e_1M + e_2M \subseteq M \\ \implies M &= e_1M + e_2M \end{aligned}$$

Let $m \in e_1M \cap e_2M$, that is, $m = e_1m' = e_2m''$. Then

$$\begin{aligned} e_1m &= e_1(e_1m') \\ &= (e_1e_1)m' \\ &= e_1m' = m \\ &= e_1(e_2m'') = \underbrace{(e_1e_2)}_{=0}m'' = 0 \cdot m'' = 0 \end{aligned}$$

$\implies m = 0$. Hence $e_1M \cap e_2M = (0)$.
$\implies M = e_1M \oplus e_2M$. □

Let $m \in M$. Then

$$\begin{aligned} e_1m &= e_1(e_1m + e_2m) \\ &= e_1^2m + (e_1e_2)m \\ &= e_1m \end{aligned}$$

and

$$\begin{aligned} e_2m &= e_2(e_1m + e_2m) = e_2m \\ \alpha m &= \alpha(e_1m + e_2m) \\ &= \alpha(e_1m) + \alpha(e_2m) \\ &= \alpha(e_1m) + 0 \\ &= \alpha m = \alpha e_1m = (e_2\alpha)e_1m \\ &= e_2(\alpha e_1m) \in e_2M \end{aligned}$$

$M \xrightarrow{\alpha\cdot -} M \rightsquigarrow$ linear map $f_\alpha \colon e_1M \xrightarrow{\alpha\cdot -} e_2M$
$M \xrightarrow{e_1\cdot -} M \rightsquigarrow$ linear map, projection $M \to e_1M$
$M \xrightarrow{e_2\cdot -} M \rightsquigarrow$ linear map, projection $M \to e_2M$

$$e_1M \xrightarrow{\alpha\cdot -} e_2M$$

is a representation of Γ over k: A vector space in each vertex and a linear map for the arrow.

Given $V \xrightarrow{f} V'$, two vector spaces V, V' over k and f a linear map. How can we construct a left $k\Gamma$-module?
From above: $M = V \oplus V'$ as a vector space. Let $m = (v, v')$, then

$$
\begin{aligned}
e_1 m &\overset{\text{def}}{=} (v, 0)\\
e_2 m &\overset{\text{def}}{=} (0, v)\\
\alpha m &\overset{\text{def}}{=} (0, f(v))
\end{aligned}
$$

<u>Check</u>: M becomes a left $k\Gamma$-module!

Definition 2.2. A *representation* (V, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k is a collection of vector spaces $\{V(i)\}_{i \in \Gamma_0}$ over k and k-linear maps $f_\alpha \colon V(i) \to V(j)$ for each arrow $\alpha \colon i \to j$ in Γ_1. (We assume that $\dim_k V(i) < \infty$ for all $i \in \Gamma_0$).

Example 2.3. $\Gamma \colon 1$. A representation of Γ over k is just a vector space over k.

Example 2.4. $\Gamma \colon 1 \xrightarrow{\alpha} 2$. Representation $V(1) \xrightarrow{f_\alpha} V(2)$. For example

$$k \xrightarrow{1} k \qquad k \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} k \qquad k^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix}} k^3$$

Example 2.5. Γ:
$$
\begin{tikzcd}
 & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\
 & 4 &
\end{tikzcd}
$$
Representation:
$$
\begin{tikzcd}
 & V(1) \arrow[dl, "f_\alpha"'] \arrow[dr, "f_\beta"] & \\
V(2) \arrow[dr, "f_\gamma"'] & & V(3) \arrow[dl, "f_\delta"] \\
 & V(4) &
\end{tikzcd}
$$

For example:

$$
\begin{tikzcd}
 & k^2 \arrow[dl, "\begin{pmatrix} 1 & 0 \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}"] & \\
k \arrow[dr, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] \\
 & k^2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
 & k &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[dr, "1"'] & & k \arrow[dl, "0"] \\
 & k &
\end{tikzcd}
$$

2.1. **Maps between representations.**

Example 2.6. $\Gamma \colon 1 \xrightarrow{\alpha} 2$, k a field.
Let $f \colon M \to N$ be a homomorphism of left $k\Gamma$-modules. Then

$$
\begin{aligned}
f(e_1 m) &= f((e_1 e_1) m)\\
&= f(e_1(e_1 m))\\
&= e_1 f(e_1 m) \in e_1 N
\end{aligned}
$$

$$\implies f|_{e_1 M} : e_1 M \to e_1 N.$$

Similarly, $f|_{e_2M}\colon e2M \to e_2N$. Furthermore,

$$\begin{aligned}\alpha f|_{e_1M}(e_1m) &= \alpha f(e_1m)\\ &= f(\alpha(e_1m))\\ &= \alpha f|_{e_1M}(e_1m)\end{aligned}$$

since $\alpha = e_2\alpha$.

Hence

$$\begin{tikzcd} e_1M \arrow[r, "f|_{e_1M}"] \arrow[d, "\alpha\cdot -"] & e_1M \arrow[d, "\alpha\cdot -"] \\ e_2M \arrow[r, "f|_{e_2M}"] & e_2M \end{tikzcd}$$

Remark 2.7. $f\begin{pmatrix}1-1\\ \text{onto}\\ \text{isom.}\end{pmatrix} \quad \Leftrightarrow \quad f|_{e_iM}\begin{pmatrix}1-1\\ \text{onto}\\ \text{isom.}\end{pmatrix}$ for all i.

Definition 2.8. Let (V, f) and (V', f') be two representations of Γ over k. A *homomorphism* $h : (V, f) \to (V', f')$ is a collection of linear maps

$$h(i)\colon V(i) \to V'(i)$$

for all $i \in \Gamma_0$, such that $\forall\alpha : i \to j \in \Gamma_1$ the following diagram commutes:

$$\begin{tikzcd} V(i) \arrow[r, "h(i)"] \arrow[d, "f_\alpha"] \arrow[dr, phantom, "\circlearrowright"] & V'(i) \arrow[d, "f'_\alpha"] \\ V(j) \arrow[r, "h(j)"] & V'(j) \end{tikzcd}$$

i.e. $f'_\alpha h(i) = h(j)f_\alpha$ for all $\alpha \in \Gamma_1$.

Note 2.9. h is a(n) isomorphism, monomorphism, epimorphism if $h(i)\colon V(i) \to V'(i)$ are all isomorphisms, monomorphisms, epimorphisms respectively.

Example 2.10. (1) $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k is a field

(a)
$$\begin{tikzcd} k \arrow[r, dashed, "a\cdot -"] \arrow[d, "1"] & k \arrow[d, "0"] \\ k \arrow[r, dashed, "0"] \arrow[d, equal] & k \arrow[d, equal] \\ (V,f) & (V',f') \end{tikzcd}$$

Here $h(1) = a \cdot -$ and $h(2) = 0$ so $h = (a \cdot -, 0)$

(b)
$$\begin{tikzcd}
k \arrow[r, dashed, "0"] \arrow[d, "1"] \arrow[dr, phantom, "\Downarrow"] & 0 \arrow[d, "0"] \\
k \arrow[r, dashed, "0"] \arrow[d, equal] & k \arrow[d, equal] \\
(V,f) & (V',f')
\end{tikzcd}$$
No non-zero homomorphisms

(c)
$$\begin{tikzcd}
k^2 \arrow[r, dashed, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0&1\\1&0\end{pmatrix}"'] & k^2 \arrow[d, "\begin{pmatrix}0&1\\1&0\end{pmatrix}"] \\
k^2 \arrow[r, dashed, "\begin{pmatrix}0&1\\1&0\end{pmatrix}"'] \arrow[d, equal] & k^2 \arrow[d, equal] \\
(V,f) & (V',f')
\end{tikzcd}$$
$h = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is an isomorphism.

(2) $\Gamma\colon$
$$\begin{tikzcd}[column sep=small]
& 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & & k \arrow[dl, "1"'] \arrow[dr, "1"] & & & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] & k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \arrow[rrr, bend right] & k \arrow[dr, "1"'] & & k \arrow[dl, "0"] \\
& 4 & & & k \arrow[d, equal] \arrow[rrr, bend right, "0"] & & & k \arrow[d, equal] & \\
& & & & (V,f) & & & (V',f') &
\end{tikzcd}$$
here we have no isomorphism between (V,f) and (V',f').

2.2. **Modules and representations.** $\Gamma = (\Gamma_0, \Gamma_1)$ - quiver, k field.

$$M \text{ left } k\Gamma\text{-module} \rightsquigarrow \begin{cases} (V,f), \text{ representation of } \Gamma \\ V(i) = e_i M \\ \text{for } \alpha\colon i \to j \in \Gamma_1, \text{ we have } f_\alpha\colon V(i) = e_i M \xrightarrow{\alpha\cdot -} e_j M = V(j) \\ f_\alpha(e_i m) = \alpha e_i m \end{cases}$$

$$(V,f) \text{ representaion of } \Gamma \rightsquigarrow \begin{cases} M = \oplus_{i\in\Gamma_0} V(i),\ k\Gamma\text{-module}^* \\ m = (v_1, v_2, \dots, v_n) \in M \\ e_i m \xrightarrow{\text{def}} = (0,\dots,0,v_i,0,\dots,0) \\ \text{for } \alpha\colon i \to j \text{ in } \Gamma_1, \text{ remember } \alpha = e_j \alpha e_i \\ \alpha m \xrightarrow{\text{def}} = (0,\dots,0,f_\alpha(v_i),0,\dots,0) \text{ with } f_\alpha(v_i) \text{ in the } j\text{-th coordinate} \end{cases}$$

*Can show: This induces a left $k\Gamma$-module structure on M (see [1, page 57]).

Example 2.11. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field.

$(V,f)\colon\ k \xrightarrow{1} k \rightsquigarrow M = k\oplus k = k^2$

$$\begin{aligned} e_1\cdot(a,b) &= (a,0)\\ e_2\cdot(a,b) &= (0,b)\\ \alpha\cdot(a,b) &= (0,a)\end{aligned}$$

Note: $k\Gamma e_1 = k\{e_1,\alpha\}$. Define $\varphi\colon M\to k\Gamma e_1$ by letting

$$\varphi(1,0)=e_1 \text{ and } \varphi(0,1)=\alpha.$$

Have:

$$\begin{aligned}\alpha\varphi(a,b) = \alpha(ae_1+b\alpha) &= a\underbrace{\alpha e_1}_{=\alpha}+b\underbrace{\alpha^2}_{=0}\\ &= a\alpha\\ &= \varphi(0,a)=\varphi(\alpha(a,b))\end{aligned}$$

Similarly, $e_i\varphi(a,b)=\varphi(e_i(a,b))$. This implies that φ is a $k\Gamma$-homomorphism.

$$\left.\begin{array}{r}\operatorname{Ker}\varphi=(0)\\ \operatorname{Im}\varphi=k\Gamma e_1\end{array}\right\}\Rightarrow M\simeq k\Gamma e_1 \text{ as a left } k\Gamma\text{-module.}$$

Example 2.12. Γ:

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$

, k field.

(V,f):

$$\begin{tikzcd} & k \arrow[dl, "f_\alpha=1"'] \arrow[dr, "f_\beta=1"] & \\ k \arrow[dr, "f_\gamma=\binom{1}{0}"'] & & k \arrow[dl, "f_\delta=\binom{0}{1}"] \\ & k^2 & \end{tikzcd}$$

$M = V(1)\oplus V(2)\oplus V(3)\oplus V(4) = k\oplus k\oplus k\oplus k^2.$

$$\begin{aligned}\alpha(v_1,v_2,v_3,v_4) &= (0,v_1,0,0)\\ \gamma(v_1,v_2,v_3,v_4) &= (0,0,0,(v_2,0))\\ \gamma\alpha(v_1,v_2,v_3,v_4) &= (0,0,0,(v_1,0))\end{aligned}$$

Exercise 2.13. Show that $M\simeq k\Gamma e_1$ as a left $k\Gamma$-module.

2.3. **Special representations.**

- Zero representation: $\begin{cases} V(i)=(0), \text{ for all } i\in\Gamma_0,\\ f_\alpha=0, \text{ for all } \alpha\in\Gamma_1.\end{cases}$
- For each $i\in\Gamma_0$, we have a representation T_i given by $T_i(j)=\begin{cases}k, \text{ if } j=i\\ (0), \text{ otherwise}\end{cases}$

  and $f_\alpha=0$ for all $\alpha\in\Gamma_1$.

  T_i corresponds to a left Γ-module S_i:

$S_i \simeq k$ as a vector space and $e_j v = \begin{cases} v, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$ and $\alpha v = 0$ for all $\alpha \in \Gamma_1$.

Recall 2.14. Λ k-algebra, k field.

$$\begin{array}{lcl} M \text{ left } \Lambda\text{-module} & \Rightarrow & M \ k\text{-vector space} \\ \cup & & \\ N \text{ submodule} & \Rightarrow & N \subseteq M \text{ subspace} \end{array}$$

Note 2.15. $\dim_k S_i = 1 \Rightarrow S_i$ is a simple $k\Gamma$-module.

Definition 2.16. Λ ring, $(0) \neq M$ left Λ-module. The module M is *indecomposable* if

$$M \simeq M_1 \oplus M_2$$

implies that $M_1 = (0)$ or $M_2 = (0)$.

Definition 2.17. Let $V = (V, f)$ and $V' = (V', f')$ be two representations of a quiver Γ. Define $W = (W, h) = V \oplus V'$, *the direct sum of the representations* V and V' by

$$W(i) = V(i) \oplus V'(i)$$

and

$$h_\alpha = f_\alpha \oplus f'_\alpha \colon W(i) = V(i) \oplus V'(i) \to V(j) \oplus V'(j) = W(j)$$

for all $i \in \Gamma_0$ and for all $\alpha \in \Gamma_1$.

Definition 2.18. $(0) \neq V = (V, f)$ is an *indecomposable* representation if

$$V = V_1 \oplus V_2$$

implies that $V_1 = (0)$ or $V_2 = (0)$.

Example 2.19. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field.

- $k^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} 2 \ \simeq (\ k \xrightarrow{1} k\ ) \oplus (\ k \xrightarrow{1} k\ )$
- $k \xrightarrow{1} k$ indecomposable? Others?

2.4. **Subrepresentations.** $\Gamma = (\Gamma_0, \Gamma_1)$ quiver, k field.

M, N $k\Gamma$-modules, $N \subseteq M$ submodule.

$\Rightarrow e_i N \subseteq e_i M$ subspace.

Given $\alpha\colon i \to j \in \Gamma_1$, the following diagram commutes

$$\begin{tikzcd} e_i N \arrow[r, hook] \arrow[d, "\alpha\cdot -"'] & e_i M \arrow[d, "\alpha\cdot -"] \\ e_j N \arrow[r, hook] & e_j M \end{tikzcd}$$

Definition 2.20. (a) $(V, f) \subseteq (V', f')$ is a *subrepresentation* if

(i) $V(i) \subseteq V'(i)$ subspace for all $i \in \Gamma_0$,

(ii)

$$\begin{tikzcd} V(i) \arrow[r, hook] \arrow[d, "f_\alpha"'] & V'(i) \arrow[d, "f'_\alpha"] \\ V(j) \arrow[r, hook] & V'(j) \end{tikzcd}$$

for $\alpha\colon i \to j \in \Gamma_1$, that is, $f_\alpha = f'_\alpha|_{V(i)}$.

(b) If $(V, f) \subseteq (V', f')$ is a subrepresentation, then the *factor representation* $W = (W, f'')$ of V and V' is given as
   (i) $W(i) = V'(i)/V(i)$,
   (ii)

$$\begin{tikzcd}
V(i) \arrow[r, hook] \arrow[d, "f_\alpha"'] & V'(i) \arrow[r] \arrow[d, "f'_\alpha"] & V'(i)/V(i) = W(i) \arrow[d, dashed, "f''_\alpha"] \\
V(j) \arrow[r, hook] & V'(j) \arrow[r] & V'(j)/V(j) = W(j)
\end{tikzcd}$$

where $f''_\alpha(v' + V(i)) = f'_\alpha(v') + V(j)$ for $\alpha\colon i \to j \in \Gamma_1$.

Check:

(i) f''_α is well-defined.
(ii) (W, f'') is a representation of Γ over k.
(iii) We have

$$\begin{tikzcd}
(V, f) \arrow[r, hook] \arrow[d, rightsquigarrow] & (V', f') \arrow[d, rightsquigarrow] & W = V'/V \arrow[d, rightsquigarrow] \\
M_V \arrow[r, hook] & M_{V'} & M_W \simeq M_{V'}/M_V
\end{tikzcd}$$

Definition 2.21. Λ finite dimensional k-algebra, k field. Then Λ is of *finite representation type* if there is only a finite number of non-isomorphic indecomposable finitely generated left Λ-modules.

Example 2.22. $\Lambda = k$. The only indecomposable Λ-module is k.

Example 2.23. $\Gamma\colon\ 1 \xrightarrow{\alpha} 2$, k field.

The indecomposable left $k\Gamma$-modules $\leftrightsquigarrow$ The indecomposable representations of Γ over k.

Let $(V, f) = V_1 \xrightarrow{f} V_2$ is an indecomposable representation of Γ over k.

Know: $h_1\colon V_1 \to \operatorname{Im} f \oplus \operatorname{Ker} f$

In particular, $V \underset{\exists f'}{\overset{f}{\rightleftarrows}} \operatorname{Im} f$ such that $ff' = 1_{\operatorname{Im} f}$.

$h_1\colon V_1 \to \operatorname{Im} f \oplus \operatorname{Ker} f$ given by $v \mapsto (f(v), v - f'f(v))$.

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "h_1"'] & V_2 \arrow[d, "h_2=1_{V_2}"] \\ \operatorname{Im} f \oplus \operatorname{Ker} f \arrow[r, "(\nu,0)"] & V_2 \end{tikzcd}$$

$$\simeq$$

$$\begin{tikzcd} \operatorname{Im} f \arrow[r, hook, "\nu"] & V_2 \end{tikzcd} \qquad = (0)\ \text{(ii)}$$

$$\oplus$$

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, "0"] & 0 \end{tikzcd} \qquad = (0)\ \text{(i)}$$

where $\nu\colon \operatorname{Im} f \hookrightarrow V_2$.

Case (i):

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, "0"] \arrow[d, "\simeq"] & 0 \arrow[d, equal] \\ k^t \arrow[r, "0"] & 0 \end{tikzcd} \simeq \ (k \xrightarrow{0} 0)^t$$

(V, f) indecomposable $\Rightarrow t = 1$ and $(V, f) \simeq\ k \xrightarrow{0} 0$

Case (ii): $\operatorname{Im} f \hookrightarrow V_2$

Know: $V_2 = \operatorname{Im} f \oplus V_2'$

$$\begin{tikzcd} \operatorname{Im} f \arrow[r, hook] \arrow[d, equal] & V_2 \arrow[d, equal] \\ \operatorname{Im} f \arrow[r, "\left(\begin{smallmatrix} 1_{\operatorname{Im} f} \\ 0 \end{smallmatrix}\right)"] & \operatorname{Im} f \oplus V_2' \end{tikzcd}$$

$$\simeq$$

$$\begin{tikzcd} \operatorname{Im} f \arrow[r, equal] & \operatorname{Im} f \end{tikzcd} \qquad = (0)\ \text{(i)}$$

$$\oplus$$

$$\begin{tikzcd} 0 \arrow[r, "0"] & V_2' \end{tikzcd} \qquad = (0)\ \text{(ii)}$$

Case (i):

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, equal] & V_2' \arrow[d, "\simeq"] & \simeq \ (0 \xrightarrow{0} k)^t \\ 0 \arrow[r, "0"] & k^t \end{tikzcd}$$

(V, f) indecomposable $\Rightarrow t = 1$ and $(V, f) \simeq \ 0 \xrightarrow{0} k$

Case (ii):

$$\begin{tikzcd} \operatorname{Im} f \arrow[r, "1_{\operatorname{Im} f}"] \arrow[d, "\varphi"] & \operatorname{Im} f \arrow[d, "\varphi"] & \simeq \ (k \xrightarrow{1} k)^t \\ k^t \arrow[r, "1_{k^t}"] & k^t \end{tikzcd}$$

(V, f) indecomposable $\Rightarrow t = 1$ and $(V, f) \simeq \ 0 \xrightarrow{0} k$

Check: $k \xrightarrow{1} k$, $k \xrightarrow{0} 0$ and $0 \xrightarrow{0} k$ are indecomposable.

Hence: The only indecomposable representations are the ones above

" $\Rightarrow$ " The only indecomposable left $k\Gamma$-modules are $k\Gamma e_1$, $k\Gamma e_1/\langle \alpha e_1 \rangle$ and $S_2 = k\Gamma e_2$.

$\Rightarrow k\Gamma$ is of finite representation type.

Theorem 4. *k field,* $\operatorname{char} k = p$*, G finite group with $p \mid |G|$. Then,*
kG of finite representation type $\Leftrightarrow$ All p-Sylow subgroups of G are cyclic.

Theorem 5. *Γ connected quiver without oriented cycles, k field.*
$k\Gamma$ is of finite representation type $\Leftrightarrow$ The underlying graph of Γ is a Dynkin diagram.

$$\mathbb{A}_n: \quad 1 \text{ --- } 2 \text{ --- } \cdots \quad \text{ --- } n-1 \text{ --- } n$$

$$\mathbb{D}_n: \quad \begin{tikzcd}[arrows=dash] 1 \arrow[dr] & & & & & \\ & 3 \arrow[r] & \cdots & {} \arrow[r] & n-1 \arrow[r] & n \\ 2 \arrow[ur] & & & & & \end{tikzcd}$$

$$\mathbb{E}_6: \quad \begin{tikzcd}[arrows=dash] & & 4 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 5 \arrow[r] & 6 \end{tikzcd}$$

$$\mathbb{E}_7: \quad \begin{tikzcd}[arrows=dash] & & 4 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

$$\mathbb{E}_6: \quad \begin{tikzcd}[arrows=dash] & & 4 \arrow[d] & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \end{tikzcd}$$

3. Quiver with relations

Can all algebras over a field k be represented as $k\Gamma$?
No: $\Lambda = k[x]/\langle x^2\rangle \ncong k\Gamma$ for all quivers Γ.
Have: $\dim_k \Lambda = 2$ and Λ is not semisimple.
Assume that $\Lambda \cong k\Gamma$.
$2 = \dim_k k\Gamma \geq \#$ vertices in Γ.
$\Gamma\colon\ 1 \qquad 2 \implies k\Gamma$ is semisimple Ψ
$\implies \Gamma$ has one vertex, $1 \circlearrowright \alpha \implies \dim_k k\Gamma = \infty\ \Psi$

But, $\Gamma\colon\ 1 \circlearrowright \alpha$, $\Lambda \simeq \frac{k\Gamma}{\langle \alpha^2\rangle}$

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, k field.

Definition 3.1.

(a) A relation σ in the quiver Γ over k is a k-linear combination of paths

$$\sigma = a_1 p_1 + a_2 p_2 + \cdots + a_t p_t$$

where $a_i \in k$, $e(p_i) = e(p_1)$ and $s(p_i) = s(p_1)$ for all i, and $l(p_i) \geq 2$ (the length of the path p_i)

(b) if $\varrho = \{\sigma\}_{l \in T}$ is a set of relations in Γ over k, then (Γ, ϱ) is a quiver with relations over k.

Example 3.2.

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \Gamma\colon 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$

k field, $\sigma = \gamma\alpha - \delta\beta$.

$\Gamma = \frac{k\Gamma}{\langle\sigma\rangle}$. Let M be a left $k\Lambda$-module. Any left Λ-module is a left $k\Lambda$-module, since $k\Gamma \xrightarrow{\pi} \frac{k\Gamma}{\sigma\rangle} = \Lambda$.
$\implies M$ gives rise to a representation of Γ.

$$\begin{tikzcd} & e_1M \arrow[dl, "f_\alpha = \alpha\cdot -"'] \arrow[dr, "\beta\cdot - = f_\beta"] & \\ e_2M \arrow[dr, "f_\gamma = \gamma\cdot -"'] & & e_3M \arrow[dl, "\delta\cdot - = f_\delta"] \\ & 4 & \end{tikzcd}$$

$\sigma \in k\Gamma, m \in M$.
$\sigma \cdot m \overset{\text{def}}{=} \pi(\sigma)\cdot m = 0 \cdot m = 0, \forall m \in M$
$m = e_1 m + e_2 m + e_3 m + e_4 m$, $sigma = e_4 \sigma e_1$
$0 = \sigma \cdot m = (\gamma\alpha - \delta\beta)e_1 m = \gamma(\alpha e_1 m) - \delta(\beta e_1 m) = f_\gamma f_\alpha(e_1 m) - f_\sigma f_\beta(e_1 m) =$

$\underbrace{f_\gamma f_\alpha - f_\sigma f_\beta}_{f_\sigma}(e_i m) \implies f_\sigma(e_1 M) = 0 \implies f_\sigma = 0.$

Hence Λ-module M corresponds to a representation of Γ satisfying the relation $\sigma (f_\sigma = 0)$.
Conversely, we claim that a representation (V, f) of Γ such that

$$f_\sigma = f_\gamma f_\alpha - f_\sigma f_\beta = 0$$

gives a module over Λ.

<u>Recall:</u> $I \subseteq R$ ideal: $\frac{R}{I}$-module M is the same as an R-module M such that $I \cdot M = (0)$.

$$M = V(1) \oplus V(2) \oplus V(3) \oplus V(4) \leftarrow k\Gamma\text{-module}$$
$$e_1 \cdot (v_1, v_2, v_3, v_4) = (v_1, 0, 0, 0)$$
$$\alpha \cdot (v_1, v_2, v_3, v_4) = (0, f_\alpha(v_1), 0, 0)$$

$\sigma \cdot (v_1, v_2, v_3, v_4) = (\gamma\alpha - \delta\beta) \cdot (v_1, v_2, v_3, v_4) = (0, 0, 0, f_\gamma f_\alpha(v_1) - f_\sigma f_\beta(v_1)) = (0, 0, 0, (f_\gamma f_\alpha - f_\sigma f_\beta)(v_1)) = (0, 0, 0, 0)$

$\implies M$ is a Λ-module ($\Lambda = \frac{k\Gamma}{\langle\sigma\rangle}$).

Example 3.3. Γ: $1 \circlearrowright \alpha$, $P = \{\alpha^2\}$, k field. $\Lambda = \frac{k\Gamma}{\langle\alpha^2\rangle}$. Find all induced Λ-modules.
M left Λ-module $\rightsquigarrow (V, f)$ representation of Γ satisfying the relation α^2,i.e. $f_{\alpha^2} = (f_\alpha)^2$

$V \circlearrowright f_\alpha$, $(f_\alpha)^2 = 0$

$\implies$ The minimal polynomial of f_α is x or x^2
$\implies$ The ivariant factor of f_α is x or x^2
$\implies$ The matrix of f_α is similar to a direct sum of companion matrices of x or x^2, $M_{(x)} = 0$ and $M_{(x^2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Let T be the matrix of f_α w.r.t some basis β.
Then $\exists$ an invertible matrix P such that

$$T = P \underbrace{\begin{pmatrix} r\left\{\begin{pmatrix} 0\cdots 0 \\ \vdots \ddots \vdots \\ 0 \cdots 0 \end{pmatrix}\right. & \overbrace{0}^{2s} \\ 0 & \begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) & & & 0 \\ & \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \end{pmatrix} \end{pmatrix}}_{T_0} P^{-1} \implies TP = PT_0$$

$V \circlearrowright T \iff V \circlearrowright T_0 \simeq (\, k \circlearrowright 0\,)^r \oplus (\, k^2 \circlearrowright \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\,)^s$ isomorphisme of representation

<u>Show:</u> $k \circlearrowright 0 \iff \frac{k\Gamma}{\langle\alpha\rangle}$ and $k^2 \circlearrowright \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \iff \frac{k\Gamma}{\langle\alpha^2\rangle} \implies \Lambda$ is of finite representation type.

3.1. **Finite length.** Λ ring, A a (left) Λ-module.

Definition 3.4. A has *finite length* if there exists a finite filtration.

$$\mathscr{F}: A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots A_{n-1} \supseteq A_n \supseteq A_{n+1} = (0)$$

of submodules of A such that $\frac{A_i}{A_{i+1}} = 0$ or simple for i$= 0, 1, \cdots, n$.. $\mathscr{F}$ is a *generalized composition series* of A, and if $\frac{A_i}{A_{i+1}} \neq 0$ for all i, then $\mathscr{F}$ is a *composition series*. If $S = \frac{A_i}{A_{i+1}} \neq 0$, then S is called a *composition factor of A*

Let S be a simple Λ-module. Let

$$m_s^{\mathscr{F}}(A) \overset{\text{def}}{=} | \{i!!!!!!! \frac{A_i}{A_{i+1}} \simeq S\} | \ , l_{\mathscr{F}} \overset{\text{def}}{=} \sum_{\substack{[S]\text{isomorphism} \\ \text{classes of simples}}} m_s^{\mathscr{F}}(A) \text{ and } l(A) \overset{\text{def}}{=} \min_{\substack{\mathscr{F}\text{generalized} \\ \text{composition series}}} l_{\mathscr{F}}(A)$$

Example 3.5. (1) Λ ring, S simple Λ-module.
Composition series: $S \supseteq (0)$
composition factors: $\{S\}$
$\implies m_T(S) = \left\{\begin{matrix} 1 & \text{if } T \simeq S \\ 0 & \text{otherwise} \end{matrix}\right\} \implies l(S) = 1$

(2) $\Lambda = k[x]$, $f(x)$ irredicible
$S_f = \frac{k[x]}{\langle f(x)\rangle}$ - simple Λ- module.
$\implies l(S_f) = 1$, while $\dim_k S_f = \deg f(x)$

(3) $\Gamma\colon\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta}$, k field , $\Lambda = k\Gamma$

$$\begin{tikzcd}
 & k \arrow[d, "1"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] \\
\mathscr{F}: & k \arrow[d, "1"] & k \arrow[l, "1"'] \arrow[d, "1"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] \\
 & k & k & k & 0 \\
 & \| & \| & \| & \| \\
 & V_o \arrow[d, leftrightsquigarrow] & V_1 \arrow[d, leftrightsquigarrow] & V_2 \arrow[d, leftrightsquigarrow] & V_3 \arrow[d, leftrightsquigarrow] \\
M \quad = & M_0 \ \supseteq & M_1 \ \supseteq & M_2 \ \supseteq & M_3 = (0)
\end{tikzcd}$$

$$\frac{V_0}{V_1} \simeq \begin{tikzcd} k \arrow[d] \\ 0 \arrow[r, squiggly, leftrightarrow] \arrow[d] & S_1 \\ 0 \end{tikzcd}, \quad \frac{V_1}{V_2} \simeq \begin{tikzcd} k \arrow[d] \\ 0 \arrow[r, squiggly, leftrightarrow] \arrow[d] & S_2 \\ 0 \end{tikzcd}, \quad \frac{V_2}{V_3} \simeq \begin{tikzcd} k \arrow[d] \\ 0 \arrow[r, squiggly, leftrightarrow] \arrow[d] & S_3 \\ 0 \end{tikzcd}$$

$\implies l_{\mathscr{F}}(M) = 3$

(4) Γ $\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 & & 3 \end{tikzcd}$, k field, $M = k\Gamma e_1 \iff$ $\begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\ k & & k \end{tikzcd}$

need more !!!!!!!!!!!!!!!!!!!!

Note 3.6. (1) Composition serice are not unique!
(2) $l_{\mathscr{F}}(M) = l_{\mathscr{G}}(M)$
(3) The set of composition factors is the same for $\mathscr{F}$ and $\mathscr{G}$

The proof of Jordan-Hlder theorem goes bu induction on length and using short exact sequences.

Definition 3.7. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a *text*((short) exact sequence) of (left) Λ-module if

(i) f is injective (1-1)
(ii) g is surjective (onto)
(iii) Im(f) = ker(g)

Note 3.8. (1) $A \supseteq B\Lambda$-modules, submodule then

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence

(2) If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence, then
(a) $C = \mathrm{Im}(g) \simeq \frac{B}{\mathrm{Ker(g)}} \simeq \frac{B}{\mathrm{Im(f)}}, \mathrm{Im}(f) \simeq A$.
(b) $B = (0) \implies A = (0)$ and $C = (0)$.

Example 3.9. (1) $0 \longrightarrow \mathbb{Z} \xrightarrow{-\cdot n} \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{n\mathbb{Z}} \longrightarrow 0$ exact

(2) M, N Λ-modules. $0 \longrightarrow M \xrightarrow{\binom{1}{0}} M \oplus N \xrightarrow{(1\ 0)} N$ $\quad O$ exact

$$m \longmapsto (m, 0)$$

$$(m, n) \longmapsto n$$

(3) $\Lambda = \frac{k\Gamma}{\langle p \rangle}$, $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ exact sequence of Λ-modules.

$$\begin{tikzcd}
 & V_A(i) \arrow[r, "f|_{V_A(i)}"] \arrow[d, equal] & V_B(i) \arrow[r] \arrow[d, equal] & V_C(i) \arrow[d, equal] & \\
0 \arrow[r] & e_iA \arrow[r, "f|_{e_iA}"] & e_iB \arrow[r, "g|_{e_iB}"] & e_iC \arrow[r] & 0
\end{tikzcd}$$

exact sequence for all i

Hence, $0 \longrightarrow (V', f') \xrightarrow{g} (V, f) \xrightarrow{h} (V'', f'') \longrightarrow 0$ is an exact sequence of representation if $0 \longrightarrow V'(i) \xrightarrow{g(i)} V(i) \xrightarrow{h(i)} V''(i) \longrightarrow 0$ is exact for all i $\in \Gamma_0$

Exercise 3.10. $f\colon\ A \longrightarrow B$ and $g\colon\ B \longrightarrow C$, Λ-homomorphisme, $B' \subseteq B$ submodule.
(1) $f^{-1}(B') = \{a \in A \mid f(a) \in B'\} \subseteq A$ submodule
(2) $g(B') = \{g(b') \mid b' \in B'\} \subseteq C$ submodule

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence and let $\mathscr{F}$ be a generalized composition series of B.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r] \arrow[d, equal] & 0 \\
 & A_o = f^{-1}(B_0) & B_0 & g(B_0) = C_0 & \\[-15pt]
 & \cup\!\mid & \cup\!\mid & \cup\!\mid & \\[-15pt]
 & A_1 = f^{-1}(B_1) & B_1 & g(B_1) = C_1 & \\[-15pt]
 & \cup\!\mid & \cup\!\mid & \cup\!\mid & \\[-15pt]
 & A_2 = f^{-1}(B_2) & B_2 & g(B_2) = C_2 & \\[-15pt]
 & \cup\!\mid & \cup\!\mid & \cup\!\mid & \\[-15pt]
 & \vdots & \vdots & \vdots & \\[-15pt]
 & \cup\!\mid & \cup\!\mid & \cup\!\mid & \\[-15pt]
 & A_n = f^{-1}(B_n) = (0) & B_n = (0) & g(B_n) = C_n = (0) & \\[-15pt]
 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\[-15pt]
 & \mathscr{F}' & \mathscr{F} & \mathscr{F}'' &
\end{tikzcd}$$

Proposition 6. *(a) $\mathscr{F}'$ is generalized composition series of A.*
$\mathscr{F}''$ is generalized composition series of C.

(b) $m_S^{\mathscr{F}}(B) = m_S^{\mathscr{F}'}(A) + m_S^{\mathscr{F}''}(C) \forall S$ simple

Proposition 7. *Given a Λ-module A of finite length and a Λ-homomorphism f : $A \mapsto A$. The following are equivalent*
(a) f is an isomorphism

(b) f is a monomorphism (1-1)
(c) f is an epimorphism (onto)

Proof. Clearly $(a) \implies (b)$ and $(a) \implies (c)$ by def. We have the exact sequence

$$0 \to f(A) \hookrightarrow A \to A/f(A) \to 0$$

$\underline{(a) \implies (b)}$: f 1-1 $\implies A \simeq f(A) \implies l(A) = l(f(A)) \implies l(A/f(A)) = 0 \implies f(A) = A \implies f$ onto.

$\underline{(c) \implies (a)}$: f onto $\implies f(A) = A \implies l(A/f(A)) = 0 \implies l(A) = l(f(A))$
$c \to \operatorname{Ker} f \hookrightarrow A \to f(A) \to 0$ exact $\implies l(\operatorname{Ker} f) = 0 \implies \operatorname{Ker} f = (0) \implies f$ 1-1 $\implies f$ isomorphism. □

Remark 3.11. The proof of 7 holds for all $f : A \mapsto B$ with $l(A) = l(B)$, so if $l(A) = l(B)$ and $f : A \mapsto B$, then

$$f \text{ isomorphism } \Leftrightarrow f \text{ 1-1 } \Leftrightarrow f \text{ onto.}$$

Recall: $0 \to A \to B \to C \to 0$ exact and $l(B) < \infty$, then $l(A)$ and $l(C)$ are finite too.

Proposition 8. *If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact and A and C have finite length, then also B has finite length and $l(B) = l(A) + l(C)$.*

Proof. Let

$$\mathcal{F}' : A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{n-1} \supseteq A_n = (0)$$

and

$$\mathcal{F}'' : C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{m-1} \supseteq C_m = (0)$$

be two comp. series of A and C, respectivley. Consider the following chain of submods of B:

$$\mathcal{F} : B = g^{-1}(C) \supseteq g^{-1}(C_1) \supseteq g^{-1}(C_2) \supseteq \cdots \supseteq g^{-1}(C_m) = \operatorname{Ker} g$$

and

$$\operatorname{Ker} g = \operatorname{Im} f = f(A) \subseteq f(A_1) \subseteq f(A_2) \subseteq \cdots \subseteq f(A_{n-1}) \subseteq f(A_n) = (0) \tag{3.1}$$

We want to show that $\mathcal{F}$ is a comp. series of B. Let $g_i = g|_{g^{-1}C_i} : g^{-1}(C_i) \mapsto C_i\big(b_i \mapsto g(b_i)\big)$. Then g_i is clearly surjective (since g is). The composition

$$\Psi_i : \Pi_i g_i : g^{-1}(C_i) \longrightarrow C_i \xrightarrow{\Pi_i} C_i/C_{i+1}$$

$$c_i \mapsto c_i + C_{i+1}$$

onto (comp. of two onto maps) and we have $b_i \in \operatorname{Ker} \Psi_i \Leftrightarrow \Pi_i(g_i(b_i)) = 0$ in $C_i/C_{i+1} \Leftrightarrow g(b_i) + C_{i+1} = 0$ so $g(b_i)) \in C_{i+1} \Leftrightarrow b_i \in g^{-1}(C_{i+1}) \implies \operatorname{Ker} \Psi_i = g^{-1}/C_{i+1}) \implies g^{-1}(C_i)/g^{-1}(C_{i+1}) \simeq \operatorname{Im} \Psi_i$ since $g^{-1}(C_{i+1}) = \operatorname{Ker} \Psi_i$ and C_i/C_{i+1} is simple by definition of $\mathcal{F}''$ □

Let $f_i = f|_{A_i} : A_i \to f(A_i$ (i.e. $a_i \mapsto f(a_i)$) which clearly is onto. The composition

$$\theta_i = p_i f_i : A_i \xrightarrow{f_i} f(A_i) \xrightarrow{p_i} f(A_i)/f(A_{i+1})$$

where $p_i(b_i) = b_i + f(A_{i+1})$, is onto (composition of two onto maps) and we have

$$\begin{aligned} a_i \in \operatorname{Ker}\theta_i &\Leftrightarrow p_i f_i(a_i) = 0 \text{ in } f(A_i)/f(A_{i+1}) \\ &\Leftrightarrow f(a_i) + f(A_{i+1}) = 0 \\ &\Leftrightarrow f(a_i) \in f(A_{i+1}) \\ \exists a_{i+1} \in A_{i+1} &\text{ such that } f(a_{i+1} = f(a_i) \text{ (because } f 1-1) \\ &\Leftrightarrow a_i = a_{i+1} \\ &\Leftrightarrow a_i \in A_{i+1} \\ &\implies \operatorname{Ker}\theta_i = A_{i+1} \\ &\implies A_i/\operatorname{Ker}\theta_i \simeq \operatorname{Im}\theta_i = f(A_i)/f(A_{i+1}) \end{aligned}$$

$A_i/\operatorname{Ker}\theta_i = A_i/A_{i+1}$ is simple by definition of $\mathcal{F}'$. Hence $\mathcal{F}$ is a composition series of B and

$$l(B) = l(A) + l(C)$$

Definition 3.12. A collection $\mathcal{C}$ of modules (a full subcategory) is <u>closed under extensions</u> if for each exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with $A, C \in \mathcal{C}$, then B is also in $\mathcal{C}$

Let $fl(\Lambda$ be the collection of Λ-modules of finite length.

Proposition 9. *(a) $fl(\Lambda)$ is closed under extensions and contains the simples. Furthermore, $fl(\Lambda)$ is closed under submodules and factor modules.*
(b) Let $\mathcal{C}$ be a collection of Λ-modules that is closed under extensions and contains the simple Λ-modules. Then $fl(\Lambda) \subseteq \mathcal{C}$

Proof. (a): [FIX REFERENCES]
(b) Let $B \in fl(\Lambda)$ with $l(B) = n$. Induction on n.
$n = 1$: Then B is simple and $B \in \mathcal{C}$.
$n > 1$: Choose $0 \subsetneq A \subsetneq B$ submodule. This is possible since B is not simple. Then

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact with $l(A), l(B/A) < l(B) = n$. Induction $\implies$ $A, B/A \in \mathcal{C}$. $\mathcal{C}$ is closed under extensions so $B \in \mathcal{C}$ and hence $fl(\Lambda) \subseteq \mathcal{C}$. □

<u>Recall:</u> A module M is Noetherian (Artinian) if for every ascending (descending) chain of submodules of M:

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots \subseteq M$$
$$(M \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \dots)$$

$\exists n$ such that $M_n = M_{n+1} = \dots$.
M is Noetherian (Artinian) if, and only if, every non-empty set of sub-modules of M has a maximal (minimal) element.

Proposition 10. *A Λ-module. $l(A) < \infty \Leftrightarrow A$ is Artinian and Noetherian.*

Proof. $\underline{\implies:}$ Want to show: $fl(\Lambda) \subseteq \mathrm{art}(\Lambda)$ = collection of Artinian Λ-modules. Will use Proposition 9 (b). Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be exact. We claim that $A, C \in \mathrm{art}(\Lambda) \implies B \in \mathrm{art}(\Lambda)$. Consider a descending chain $B = B_0 \subseteq B_1 \subseteq \dots$. As before we get induced chains of submodules in A and C.

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_i \supseteq \dots \quad \text{with } A_i = f^{-1}(B_i)$$
$$C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_i \supseteq \dots \quad \text{with } C_i = g(B_i)$$

and induced exact squence

$$0 \longrightarrow A_i/A_{i+1} \xrightarrow{\bar{f}} B_i/B_{i+1} \xrightarrow{\bar{g}} C_i/C-i+1 \longrightarrow 0$$

since $A, C \in \mathrm{art}(\Lambda)$ there exists N such that

$$A_i = A_{i+1} \text{ and } C_i = C_{i+1} \quad i \geq N$$
$$\implies A_i/A_{i+1} = (0) \text{ and } C_i/C_{i+1} = (0) \quad i \geq N$$
$$\implies B_i/B_{i+1} = 0 \implies B_i = B_{i+1} \quad i \geq N$$
$$\implies B \in \mathrm{art}(\Lambda)$$
$$\implies \mathrm{art}(\Lambda) \text{ is closed under extensions.}$$

Clear that $\mathrm{art}(\Lambda)$ contains the simple Λ-modules: Proposition 9(b) $\implies$ $\mathrm{fl}(\Lambda) \subseteq \mathrm{art}(\Lambda)$
<u>Exercise:</u> Similarly, $\mathrm{fl}(\Lambda) \subseteq \mathrm{noeth}(\Lambda)$, collection of noehterian Λ-modules. This implies that $\mathrm{fl}(\Lambda) \subseteq \mathrm{art}(\Lambda) \cap \mathrm{noeth}(\Lambda)$.

$\underline{\Leftarrow}$: Assume that $B \neq (0)$ is Artinian and Noetherian. Since B is Artinian B has a simple submodule $S \subseteq B$. ($\mathcal{F} = \{U \subseteq B \mid U \neq (0)\}$ has a minimal element). Consider $\mathcal{F}' = \{U \subseteq B \mid l(U) < \infty\}$. Then $\mathcal{F}' \neq \emptyset$ since $S \in \mathcal{F}'$. Since B is Noetherian $\mathcal{F}'$ has a maximal element $A \subseteq B$ and $A \in \mathrm{fl}(\Lambda)$. Assume that $A \subsetneq B$, i.e. $B/A \neq (0)$. B Artinian $\implies$ B/A Artinian $\implies$ $\exists T \subseteq B/A$ simple submodule. Consider the natural projection $p : B \to B/A$. Then

$$p|_{p^{-1}} : p^{-1}(T) \to T \quad (\subseteq B/A)$$

is onto and $\mathrm{Ker}\, p|_{p^{-1}} = A$. Hence $(p^{-1}(T))/A \simeq T$ and we have an exact sequence

$$0 \longrightarrow A \longrightarrow p^{-1}(T) \longrightarrow T \longrightarrow 0$$

Now $l(p^{-1}(T)) = l(A) + 1$, a contradiction and we conclude that $A = B$ and $B \in \mathrm{fl}(\Lambda)$. □

<u>Note:</u>

(1) If Λ is a ring with 1 then Λ is artinian if, and only if, $l({}_\Lambda\Lambda) < \infty$

Proof. $\underline{\Rightarrow}$: Λ left Artinian $\implies$ Λ left Noetherian $\implies$ ${}_\Lambda\Lambda \in \mathrm{art}(\Lambda) \cap \mathrm{noeth}(\Lambda) \implies l({}_\Lambda\Lambda) < \infty$
$\underline{\Leftarrow:}$ $l({}_\Lambda\Lambda) < \infty \implies {}_\Lambda\Lambda \in \mathrm{art}(\Lambda) \cap \mathrm{noeth}(\Lambda) \implies \Lambda$ left Artinian. □

(2) Λ is left Artinian. *Challenge:* $\text{fl}(\Lambda) = \text{mrd}(?)(\Lambda)$ - finitely generated Λ-modules.
(3) $\Lambda = \mathbb{Z}, M = \mathbb{Z}/(n)$
$n = p_1^{m_1} p_2^{m_2} \dots p_t^{m_t}$, p_i different primes and $m_i \geq 1$
$$m_{\mathbb{Z}/p}(M) = \begin{cases} m_i & \text{if } p = p_i \\ 0 & \text{otherwise} \end{cases}$$

Proposition 11. *Λ a ring, B semisimple Λ-module. TFAE:*

(a) $l(B) < \infty$
(b) B is Artinian.
(c) B is Noetherian

Proof. Excercise $\square$

4. Radical

Definition 4.1. Λ ring. The *(left) radical of* Λ is the left ideal

$$\mathfrak{r} = \text{rad}\,\Lambda = \cap_{\mathfrak{m}\text{maximal left ideal}} \mathfrak{m}$$

(Also called the Jacobsen radical of Λ).

Know: $\mathfrak{r}$ is a left ideal.
Show: $\mathfrak{r}$ is an ideal.

Example 4.2. If Λ is a division ring, then $\mathfrak{r} = (0)$.

Example 4.3. $\Lambda = \mathbb{Z}$, $\langle p \rangle$ – maximal ideal if p is a prime.

$$\mathfrak{r} = \cap_{p \text{ prime}} \langle p \rangle = \langle n \rangle = (0)$$

$$\langle n \rangle \subseteq \langle p \rangle \Rightarrow p \mid n, \forall p \text{ prime} \Rightarrow n = 0.$$

Example 4.4. $\Lambda = \mathbb{Q} \times \mathbb{Q}$, $\mathfrak{m}_1 = \mathbb{Q} \times (0)$, $\mathfrak{m}_2 = (0) \times \mathbb{Q}$ - both are maximal ideals.

$$(0) = \mathfrak{m}_1 \cap \mathfrak{m}_2 \subseteq \mathfrak{r} \Rightarrow \mathfrak{r} = (0).$$

In general, if we can find a finite set of maximal ideals $\{\mathfrak{m}_i\}_{i=1}^t$ such that $\cap_{i=1}^t \mathfrak{m}_i = (0)$, then $\mathfrak{r} = (0)$.

Exercise 4.5. Show that Λ semisimple $\Rightarrow \text{rad}\,\Lambda = (0)$.

Example 4.6. Γ:
$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$
, $\rho = \{\gamma\alpha - \delta\beta\}$, k a field and $\Lambda = k\Gamma/\langle \rho \rangle$.

What is $\text{rad}\,\Lambda$?

Know: $1_\Lambda = \overline{e_1} + \overline{e_2} + \overline{e_3} + \overline{e_4}$, where $\overline{e_i} \cdot \overline{e_j} = \begin{cases} \overline{e_i}, \text{ if } i = j \\ 0, i \neq j \end{cases}$

Exercise 4.7.

$$\begin{aligned}\Lambda &= \Lambda\overline{e_1} \oplus \Lambda\overline{e_2} \oplus \Lambda\overline{e_3} \oplus \Lambda\overline{e_4}\\ \mathfrak{m}_1 &= \Lambda\{\alpha,\beta\} \oplus \Lambda\overline{e_2} \oplus \Lambda\overline{e_3} \oplus \Lambda\overline{e_4}\\ \mathfrak{m}_2 &= \Lambda\overline{e_1} \oplus \Lambda\overline{\gamma} \oplus \Lambda\overline{e_3} \oplus \Lambda\overline{e_4}\\ \mathfrak{m}_3 &= \Lambda\overline{e_1} \oplus \Lambda\overline{e_2} \oplus \Lambda\overline{\delta} \oplus \Lambda\overline{e_4}\\ \mathfrak{m}_4 &= \Lambda\overline{e_1} \oplus \Lambda\overline{e_2} \oplus \Lambda\overline{e_3} \oplus (0)\end{aligned}$$

(4.1)

$$\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \cap \mathfrak{m}_4 = \langle \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta} \rangle \xrightarrow{?} \operatorname{rad}\Lambda.$$

Proposition 12. *For any ring Λ and any $\lambda \in \Lambda$, the following are equivalent.*

(i) $\lambda \in \operatorname{rad}\Lambda$,
(ii) $1 - x\lambda$ *is* left invertible *for all $x \in \Lambda$ (i.e. $\exists x' \in \Lambda$ such that $x'(1 - x\lambda) = 1$),*
(iii) $\lambda S = (0)$ *for any simple Λ-module S.*

Proof. (i) $\Rightarrow$ (ii): Suppose $\exists x \in \Lambda$ such that $1 - x\lambda$ is not left invertible with $\lambda \in \Lambda$. Then $\Lambda(1 - x\lambda)$ is a proper left ideal in Λ. Any proper left ideal is contained in a maximal left ideal. If $\lambda \in \mathfrak{m}$, then $1 \in \mathfrak{m}$, contradiction. So $\lambda \notin \mathfrak{m}$ and in particular $\lambda \notin \operatorname{rad}\Lambda$.

(ii) $\Rightarrow$ (iii): Suppose $\exists$ a simple Λ-module S such that $\lambda S \neq (0)$, i.e. $\exists 0 \neq s \in S$ with $\lambda s \neq 0$. Have $(0) \neq \Lambda(\lambda s) \subseteq S$.

S simple $\Rightarrow \Lambda\lambda s = S$.

Hence, $\exists x \in \Lambda$ such that $x\lambda s = s \Rightarrow (1 - x\lambda)s = 0$

If $1 - x\lambda$ is left invertible, then $s = 0 \Rightarrow 1 - x\lambda$ is not left invertible.

(iii) $\Rightarrow$ (i): Let $\mathfrak{m}$ be a maximal left ideal in Λ. Then $\Lambda/\mathfrak{m}$ is a simple left Λ-module. By assumption

$$\lambda \cdot \Lambda/\mathfrak{m} = (0),$$

in particular

$$\lambda(1 + \mathfrak{m}) = \lambda + \mathfrak{m} = \overline{0}$$

and $\lambda \in \mathfrak{m}$ for all maximal left ideals $\mathfrak{m}$ in Λ. Hence $\lambda \in \operatorname{rad}\Lambda$. □

Definition 4.8. Let M be a (left) Λ-module, and let

$$\operatorname{Ann}_\Lambda(M) = \{\lambda \in \Lambda \mid \lambda m = 0, \forall m \in M\}.$$

The set $\operatorname{Ann}_\Lambda(M)$ is called the *annihilator* of M.

Note: $\operatorname{Ann}_\Lambda(M)$ is a two-sided ideal in Λ.

Corollary 13. *Given a ring Λ*

$$\operatorname{rad}\Lambda = \cap_{S \textit{ simpleleft } \Lambda\textit{-module}} \operatorname{Ann}_\Lambda(S).$$

In particular, $\operatorname{rad}\Lambda$ *is a two-sided ideal in* Λ.

Proof. Follows from (i) $\Leftrightarrow$ (iii) in Proposition 12. □

Can we find $\operatorname{rad}\Lambda$ from this?
$S \simeq S' \Rightarrow \operatorname{Ann}_\Lambda(S) = \operatorname{Ann}_\Lambda(S')$.

Theorem 14 (Nakayama Lemma). *Given a ring Λ and a finitely generated Λ-module M. If $\mathfrak{a}$ is an ideal in Λ with $\mathfrak{a} \subseteq \operatorname{rad}\Lambda$, then $\mathfrak{a}M = M$ implies that $M = (0)$.*

Proof. Suppose that $M \neq (0)$ and $\mathfrak{a}M = M$. Let $\{m_1, m_2, \ldots, m_t\}$ be a minimal set of generators for M as a Λ. Since $\mathfrak{a}M = M$, we have that

$$m_1 = \sum_{i=1}^{t} \lambda_i m_i$$

for $\lambda_i \in \mathfrak{a} \subseteq \operatorname{rad}\Lambda$.

$$\Rightarrow (1 - \lambda_1)m_1 = \sum_{i=2}^{t} \lambda_i m_i$$

Since $\lambda_1 \in \mathfrak{a} \subseteq \operatorname{rad}\Lambda \overset{\text{Proposition 12}}{\Rightarrow} 1 - \lambda_1$ has a left inverse, say u.

$$\Rightarrow m_1 = u(1 - \lambda_1)m_1 = \sum_{i=2}^{t} u\lambda_i m_i$$

$$\Rightarrow \ M \text{ can be generated by } \{m_2, \ldots, m_t\}. \text{ Contradiction!}$$

If $t = 1$, then $M = (0)$. Contradiction! If $t > 1$, then we have a contradiction to the choice of generating set $\{m_1, m_2, \ldots, m_t\}$. $\Rightarrow \mathfrak{a}M \neq M$. □

Recall 4.9. A left ideal $\mathfrak{a} \subseteq \Lambda$ is *nilpotent* if $\exists n \geq 1$ such that $\mathfrak{a}^n = (0)$.

Lemma 15. Λ *ring.*

(a) *If* Λ *is a left (right) artinian, then* $\operatorname{rad}\Lambda$ *is nilpotent.*
(b) *If* $\mathfrak{a} \subseteq \Lambda$ *is a nilpotent left ideal, then* $\mathfrak{a} \subseteq \operatorname{rad}\Lambda$

Proof. (a) $\mathfrak{r} = \operatorname{rad}\Lambda$.

$$\cdots \supseteq \mathfrak{r}^i \supseteq \mathfrak{r}^{i+1} \supseteq \cdots$$

is a descending chain of left ideals in Λ.

Λ left artinian $\Rightarrow \mathfrak{r}^m = \mathfrak{r}^{m+1} = \cdots$ for some m

$$M = \mathfrak{r}^m = \mathfrak{r}^{m+1} = \mathfrak{r}\mathfrak{r}^m = \mathfrak{r}M$$

Λ left artinian $\Rightarrow \Lambda$ left noetherian.

$\mathfrak{r}^m = M \subseteq \Lambda$ left ideal $\Rightarrow M = \mathfrak{r}^m$ finitely generated Λ-module.

Nakayama Lemma $\Rightarrow \mathfrak{r}^m = (0)$ and $\operatorname{rad}\Lambda$ is nilpotent.

(b) Assume that $\mathfrak{a}$ is a nilpotent left ideal in Λ, say $\mathfrak{a}^n = (0)$ for some $n \geq 1$. Let $a \in \mathfrak{a}$. Then for all $x \in \Lambda$ we have $xa \in \mathfrak{a}$ and $(xa)^n = 0$.

$\Rightarrow (1 + (xa) + (xa)^2 + \cdots (xa)^{n-1} = 1 - (xa)^n = 1$

$\Rightarrow 1 - xa$ has a left inverse for all $x \in \Lambda$.

Proposition 12 $\Rightarrow a \in \operatorname{rad}\Lambda \Rightarrow \mathfrak{a} \subseteq \operatorname{rad}\Lambda$. □

Recall 4.10.

$$\begin{aligned} \Lambda \text{ semisimple} &\Leftrightarrow {}_\Lambda\Lambda \text{ semisimple } \Lambda\text{-module} \\ &\Leftrightarrow \Lambda \simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_t}(D_t) \\ &\qquad n_i \geq 1, D_i \text{ division ring}, t < \infty \\ &\Leftrightarrow \Lambda \text{ left artinian and has no nilpotent (left) ideals} \end{aligned}$$

Theorem 16. Λ *ring.*

Λ *semisimple* $\Leftrightarrow \Lambda$ *left artinian and* $\operatorname{rad}\Lambda = (0)$.

Proof. $\Rightarrow$

$$\Lambda \text{ semisimple} \Rightarrow \Lambda \text{ left artinian} \Rightarrow \operatorname{rad}\Lambda \text{ is nilpotent}$$

Using this we obtain:

$$\Lambda \text{ semisimple} \Rightarrow \Lambda \text{ has no non-zero nilpotent left ideals} \Rightarrow \operatorname{rad}\Lambda = (0)$$

$\Leftarrow$ Assume that Λ is left artinian with $\operatorname{rad}\Lambda = (0)$
Lemma **??** $\Rightarrow$ Λ has no non-zero nilpotent left ideals.
$\Rightarrow$ Λ is semisimple. □

Theorem 17. Λ *left artinian,* $\mathfrak{r} = \operatorname{rad}\Lambda$. *Then*

(a) $\Lambda/\mathfrak{r}$ *is a semisimple ring.*
(b) *A left* Λ*-module* M *is semisimple if and only if* $\mathfrak{r}M = (0)$.
(c) *There are only finitely many non-isomorphic simple* Λ*-modules, and they all occur as direct summands of* $\Lambda/\mathfrak{r}$.
(d) Λ *is left noetherian.*

Proof. (a) Λ left artinian $\Rightarrow$ $\Lambda/\mathfrak{r}$ left artinian.
$\operatorname{rad}(\Lambda/\mathfrak{r}) = (\operatorname{rad}\Lambda)/\mathfrak{r}$.
Theorem 16 $\Rightarrow$ $\Lambda/\mathfrak{r}$ is semisimple.
(b)–(d): Exercise, see the book Proposition 3.1 page 9. □

Recall 4.11. Λ left artinian $\Leftrightarrow$ $l({}_\Lambda\Lambda) < \infty$.

Corollary 18. Λ *ring. TFAE:*

(a) Λ *left artinian.*
(b) *Every finitely generated* Λ*-module has finite length.*
(c) $\mathfrak{r} = \operatorname{rad}\Lambda$ *is nilpotent and* $\mathfrak{r}^i/\mathfrak{r}^{i+1}$ *is finitely generated semisimple* Λ*-module for all* $i \geq 0$.

Proof. (b) $\Rightarrow$ (a): In particular, Λ as a left Λ-module has finite length.
$\Rightarrow$ Λ is left artinian (and left noetherian).
(a) $\Rightarrow$ (c): $\mathfrak{r}$ is nilpotent by Lemma **??** (a), say $\mathfrak{r}^n = (0)$.

$$\begin{aligned} \text{Theorem ?? (d)} \Rightarrow\ & \Lambda \text{ left noetherian.} \\ \Rightarrow\ & \mathfrak{r}^i \text{ finitely generated for all } i \text{ (as a left ideal)} \\ \Rightarrow\ & \mathfrak{r}^i/\mathfrak{r}^{i+1} \text{ finitely generated for all } i \text{ (as a left ideal)} \end{aligned}$$

Theorem **??** (b) $\Rightarrow$ $\mathfrak{r}^i/\mathfrak{r}^{i+1}$ semisimple Λ-module for all $i \geq 0$, since $\mathfrak{r}\cdot\mathfrak{r}^i/\mathfrak{r}^{i+1} = (0)$
(c) $\Rightarrow$ (b): Suppose $\mathfrak{r}^n = (0)$ for some $n \geq 1$. Consider: $\Lambda \supseteq \mathfrak{r} \supseteq \mathfrak{r}^2 \supseteq \mathfrak{r}^3 \supseteq \cdots \supseteq \mathfrak{r}^{n-1} \supseteq \mathfrak{r}^n = (0)$.
In particular, we have exact sequences

□

5. Radical of a module

Definition 5.1. Λ ring, $A \subseteq B$ Λ-modules.
A is small in B if $A + X = B$ implies that $X = B$ for every submodule X of B.

Example 5.2. (1) $\Lambda = \mathbb{Z}$ and $B = \Lambda$ then the only small submodule of B is (0). If $(0) \neq A \subsetneq B$ then $A = \mathbb{Z}_n$ for some $n \neq 0,\ 1$. Choose an integer $m \neq 0,\ 1$ such that $\gcd(n, m) = 1$. Then $B = \mathbb{Z}_n + \mathbb{Z}_m = A + X$ but $X \neq B$. Hence A is not small.

(2) $\Gamma:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k a field $\Lambda = k\Gamma$, $B = k\Gamma e_1$

$$
\begin{tikzcd}[row sep=small]
 & & & & & A \arrow[d, equal] & & \\
 & k \arrow[d, "1"] & & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] \\
B = k\Gamma e_1 \arrow[r, squiggly] & k \arrow[d, "1"] & \supseteq & k \arrow[d, "1"] & \supseteq & 0 \arrow[d] & \supseteq & 0 \arrow[d] \\
 & k & & k & & k & & 0
\end{tikzcd}
$$

$$
\left.\begin{array}{l}
(i)\quad A + (0) = A \neq B \\[2ex]
(ii)\quad A + A = A \neq B \\[2ex]
(iii)\quad A + \begin{tikzcd}[row sep=small] 0 \arrow[d] \\ k \arrow[d, "1"] \\ k \end{tikzcd} = \begin{tikzcd}[row sep=small] 0 \arrow[d] \\ k \arrow[d, "1"] \\ k \end{tikzcd} \neq B
\end{array}\right\} \implies A \text{ is small in } B.
$$

(3) $\Gamma:$
$$
\begin{tikzcd}
1 \arrow[d] \arrow[dr] & 4 \arrow[d] \\
2 & 3
\end{tikzcd}
, \Lambda = k\Gamma,
$$

$$
B = \begin{tikzcd} k \arrow[d, "1"'] \arrow[dr, "1"] & k \arrow[d, "1"] \\ k & k \end{tikzcd}
\supset
\begin{tikzcd} k \arrow[d, "1"'] \arrow[dr, "1"] & 0 \arrow[d] \\ k & k \end{tikzcd}
= A' \supset
\begin{tikzcd} 0 \arrow[d] \arrow[dr] & 0 \arrow[d] \\ k & 0 \end{tikzcd}
= A
$$

Exercise 5.3.
- A small in B
- A' not small in B

Definition 5.4. Λ a ring, B a Λ-module

$$\operatorname{rad} B = \bigcap_{A \text{ max. submod. of } B} A = \text{the radical of } B$$

Note 5.5. $\operatorname{rad}_\Lambda \Lambda =$ the radical of Λ as a ring.

Proposition 19. *Λ a ring, B a finitely generated Λ-module*
$A \subseteq B$ is small in $B \iff A \subseteq \operatorname{rad} B$

Proof. $\Leftarrow$: Assume $A \subseteq \operatorname{rad} B$. Let $X \subsetneq B$. WTS: $A + X \neq B$
Consider $\mathfrak{F} = \{M \mid M \subsetneq B \text{ submodule }, X \subseteq M\}$. $\mathfrak{F} \neq \emptyset$, since $X \in \mathfrak{F}$. Let $\{C_\alpha\}_{\alpha \in I}$ be a chain of submodules of B in $\mathfrak{F}$.
Let $u = \bigcup_{\alpha \in I} C_\alpha$, u is a submodule of B. If $u = B$ each element in a set of generators $\{b_1, b_2, \cdots, b_n\}$ of B must be in one C_α. Say $b_i \in C_{\alpha_i}$. The chain condition implies that $\{b_1, b_2, \cdots, b_n\} \subseteq C_\alpha$ for some $\alpha \in I \implies C_\alpha = B$.

Contradiction! $\implies$ each chain in $\mathfrak{F}$ has an upper bound in $\mathfrak{F}$.
Zorn's Lemma $\implies$ $\mathfrak{F}$ has a maximal element B_1, i.e. B_1 is a maximal submodule of B.
Then $A \subseteq \operatorname{rad} B \subseteq B_1$ and $X = B_1$, so that $A + X \subseteq B_1 \subsetneq B$
Hence A is small in B.
$\Rightarrow$: Suppose that $A \not\subseteq \operatorname{rad} B$, that is, $\exists$ maximal submodule $B_1 \subseteq B$ such that $A \not\subseteq B_1$. Then $B_1 \not\subseteq A$ $B_1 \subseteq B$, and consequently $A + B_1 = B$ (since B_1 is maximal) $B_1 \subsetneq B \implies A$ is not small in B. $\square$

Theorem 20. *Λ left artinian, A a finitely generated Λ-module. Then* $\operatorname{rad} A = \mathfrak{r}A$ *where* $\mathfrak{r} = \operatorname{rad} \Lambda$.

Proof.

$$\mathfrak{r}A \subseteq \operatorname{rad} A \tag{5.1}$$

WTS: $\mathfrak{r}A$ is small in A.
(Prop 19 $\implies$ $\mathfrak{r}A \subseteq \operatorname{rad} A$)
Let X be a submodule of A and suppose that $\mathfrak{r}A + X = A$.

$$\begin{array}{ccccccc}
\implies & \mathfrak{r}^2 A & + & \mathfrak{r}X & = & \mathfrak{r}A \\
\implies & \mathfrak{r}^2 A & + & \underbrace{\mathfrak{r}X + X}_{\parallel} & = & A \\
\implies & \mathfrak{r}^2 A & + & X & = & A
\end{array}$$

Induction: $\mathfrak{r}^n A + X = A$ for all $n \geq 1$.
Lemma **??** $\implies$ $\mathfrak{r}$ nilpotent $\implies$ $X = A$ $\implies$ $\mathfrak{r}A$ is small in A $\implies$ $\mathfrak{r}A \subseteq \operatorname{rad} A$.

$$\operatorname{rad} A \subseteq \mathfrak{r}A \tag{5.2}$$

$A/\mathfrak{r}A$ is a semisimple module since $\mathfrak{r}A/\mathfrak{r}A = (0)$ (Theroem **??** (b))
We have: $A/\mathfrak{r}A = \bigoplus_{i=1}^t S_i$, S_i simple Λ-module. Let $A_j = \bigoplus_{i=1, i\neq j}^t S_i \subseteq A/\mathfrak{r}A$, which is a maximal submodule. Furthermore $\bigcap_{j=1}^t A_j = (0)$, so that $\operatorname{rad}(A/\mathfrak{r}A) = (0)$.
5.1 $\implies$ $\mathfrak{r}A$ is contained in all maximal submodules of A
$\implies$ $\operatorname{rad}(A/\mathfrak{r}A) = (\operatorname{rad}(A) + \mathfrak{r}A)/\mathfrak{r}A = (0)$
$\implies$ $\operatorname{rad} A \subseteq \mathfrak{r}A$
$\implies$ $\operatorname{rad} A = \mathfrak{r}A$ $\square$

Example 5.6. $\Gamma:$ $$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$, $\rho = \gamma\alpha - \delta\beta, k$ field, $\Lambda = k\Gamma/\langle rho\rangle$.

We have seen: $\mathfrak{r} = \langle\overline{arrows}\rangle = \overline{J}$.

$$\begin{tikzcd} & & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \Lambda\overline{e_1} \arrow[r, rightsquigarrow] & 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & & 4 & \end{tikzcd}$$

$\text{I}\!\cup$

$$\begin{tikzcd} & & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \operatorname{rad}\Lambda\overline{e_1} = \mathfrak{r}(\Lambda\overline{e_1}) = \mathfrak{r}\overline{e_1} \arrow[r, rightsquigarrow] & 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & & 4 & \end{tikzcd}$$

6. The radical of representation

(Γ, ρ) quiver with relations, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, k field, $\Lambda = k\Gamma/\langle\rho\rangle$, $\Gamma_0 = \{1, 2, \cdots, n\}$, $\mathfrak{r} = \overline{J}$.

$$\begin{tikzcd} (V, f) \text{ representation of } (\Gamma, \rho) \arrow[r, rightsquigarrow] & M_{(V,f)} = V(1) \oplus V(2) \oplus \cdots \oplus V(n) \\ \text{I}\!\cup & \text{I}\!\cup \\ (V', f') \text{ radical of } (V, f) \arrow[r, rightsquigarrow] & \mathfrak{r}M_{(V,f)} = \{r_1m_1 + \cdots + r_tm_t \mid r_i \in \mathfrak{r}, m_i \in M_{(V,f)}\} \end{tikzcd}$$

$\mathfrak{r}$ generated by the arrows $\implies$ $\mathfrak{r}M_{(V,f)}$ is generated by elements on the form

$$\beta \cdot (v_1, v_2, \cdots v_n) = (0, \cdots, 0, \underset{\substack{\uparrow \\ s\text{-th coordinate}}}{f_\beta(v_r)}, 0, \cdots, 0) \quad \text{for } \beta : r \to s \in \Gamma_1$$

$$\implies \overline{e_s}\mathfrak{r}M_{(V,f)} = \sum_{\substack{\beta\in\Gamma_1, \\ e(\beta)=s}} \operatorname{Im} f_\beta$$

$\implies V'(i) = \overline{e_i}\mathfrak{r}M_{(V,f)} = \sum_{\substack{\beta\in\Gamma_1, \\ e(\beta)=i}} \operatorname{Im} f_\beta$ and
$f'_\alpha = f_\alpha \mid_{V'(i)} : V'(i) \to V'(j)$ for $\alpha : i \to j$.
The range is by definition OK since $\operatorname{Im} f_\alpha \subseteq V'(j)$.

Example 6.1. (1) $\Gamma:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$.

$$\begin{tikzcd} & k \arrow[d, "1"] & & 0 \arrow[d] \\ \Lambda e_1 \arrow[r, rightsquigarrow] & k \arrow[d, "1"] & \operatorname{rad}(\Lambda e_1) \arrow[r, rightsquigarrow] & k \arrow[d, "1"] \\ & k & & k \end{tikzcd}$$

$$\Lambda e_2 \rightsquigarrow \begin{array}{c} 0 \\ \downarrow \\ k \\ \downarrow{\scriptstyle 1} \\ k \end{array} \qquad \operatorname{rad}(\Lambda e_2) \rightsquigarrow \begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ k \end{array}$$

$$\Lambda e_3 \rightsquigarrow \begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ k \end{array} \qquad \operatorname{rad}(\Lambda e_3) \rightsquigarrow \begin{array}{c} 0 \\ \downarrow \\ 0 \\ \downarrow \\ 0 \end{array}$$

(2) $\Gamma:\ 1 \xrightarrow{\alpha} 2 \circlearrowright \beta$, $\rho = \{\beta^2\}, \Lambda = k\Gamma/\langle\rho\rangle$

$\Lambda e_1:\ k \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} k^2 \circlearrowright \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $\operatorname{rad}(\Lambda e_1):\ 0 \longrightarrow k^2 \circlearrowright \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$

$\Lambda e_2:\ 0 \longrightarrow k^2 \circlearrowright \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $\operatorname{rad}(\Lambda e_2):\ 0 \longrightarrow (0) \oplus k \circlearrowright 0$

Note 6.2. In general, M and N Λ-modules then $\operatorname{rad}(M \oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$

Definition 6.3. Λ left artinian, $\mathfrak{r} = \operatorname{rad}\Lambda$, A finitely generated Λ-module. Then $A/\mathfrak{r}A$ is called <u>the top of A</u>.

Example 6.4. (1) $\Gamma:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$.

$$A = \Lambda e_1 : k \xrightarrow{1} k \xrightarrow{1} k$$

$$\mathfrak{r}A : 0 \longrightarrow k \xrightarrow{1} k$$

$$A/\mathfrak{r}A : k \longrightarrow 0 \longrightarrow 0 \rightsquigarrow e_1\Lambda e_1 = ke_1$$

(2) $A = \Lambda\overline{e}_1 \rightsquigarrow k \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} k^2 \circlearrowright \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \supseteq 0 \longrightarrow k^2 \circlearrowright \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$

$A/\mathfrak{r}A = \overline{e}_1\Lambda\overline{e}_1 = k\overline{e}_1 \rightsquigarrow k \longrightarrow 0$

In general Λ left artinian, A finitely generated

$$A \to A/\mathfrak{r}A = \begin{array}{c} S_1 \\ \cup \\ x_1' \neq 0 \end{array} \oplus \cdots \oplus \begin{array}{c} S_t \\ \cup \\ x_t' \neq 0 \end{array} \quad \text{semisimple, each } S_i \text{ simple}$$

Choose $\{x_1, x_2, \cdots, x_t\}$ inverse images of x_i' in A. For $a \in A$, then $\exists \lambda_i \in \Lambda$ such that $a - \sum_{i=1}^t \lambda_i x_i \in \mathfrak{r}A$

$$a - \sum_{i=1}^{t} \lambda_i x_i = \sum_{j=1}^{n} r_j a_j, \quad r_j \in \mathfrak{r}, \quad a_j \in A$$

Let $A' = \Lambda\{x_1, \cdots, x_t\} \subseteq A$ be the submodule generated by $\{x_1, \cdots, x_t\}$ of A

$$\mathfrak{r}(A/A') = A/A'$$

Nakayama Lemma $\Longrightarrow$ $A/A' = 0$ $\Longrightarrow$ $A = A'$ generated by $\{x_1, \cdots, x_t\}$.
(or use $\mathfrak{r}^m = (0) \cdots$)

Lemma 21. *Λ left artinian, $f : A \to B$ Λ-homomorphism, A, B finitely generated Λ-modules. Then*

$$f : A \to B \text{ is onto} \iff \overline{f} : A/\mathfrak{r}A \to B/\mathfrak{r}B \text{ is onto.}$$

Proof. Let $f : A \to B$ then $f(\mathfrak{r}A) = \mathfrak{r}f(A) \subseteq \mathfrak{r}B$

$$\begin{tikzcd} a \arrow[d, mapsto] & A \arrow[r, "f"] \arrow[d, "P_A"] & B \arrow[d, "P_B"] & b \arrow[d, mapsto] \\ a + \mathfrak{r}A & A/\mathfrak{r}A \arrow[r, "\overline{f}"] & B/\mathfrak{r}B & b + \mathfrak{r} \end{tikzcd}$$

$$a + \mathfrak{r}A \longmapsto f(a) + \mathfrak{r}B$$

$\underline{\Rightarrow}$: Assume that $f : A \to B$ is onto
f, P_B onto $\implies$ $P_B \circ f = \overline{f} \circ P_A$ onto $\implies$ $\overline{f}$ onto.

$\underline{\Leftarrow}$: Assume $\overline{f} : A \to B$ is onto.
The elements of $\operatorname{Im} \overline{f}$ are $f(a) + \mathfrak{r}B$ for some $a \in A$.
$\implies$ given $b \in B$ then $\exists a \in A$ such that $b + \mathfrak{r}B = f(a) + \mathfrak{r}B$
$\implies$ $b - f(a) \in \mathfrak{r}B \implies B = \operatorname{Im} f + \mathfrak{r}B$
Since $\mathfrak{r}B = \operatorname{rad} B$ (Theorem 20) is small in B (Prop 19), we have $\operatorname{Im} f = B$ and f is onto. $\square$

Note 6.5. Only used that B was finitely generated.

Definition 6.6. $f : A \to B$ is an <u>essential epimorphism</u> if f is and epimorphism, and if $g : X \to A$ is such that $f \circ g : X \to B$ is onto then $g : X \to A$ is onto.

Example 6.7. (1) $f : A \oplus B \to A$, $f(a, b) = a$, A, B Λ-modules.
- f epi OK, f ess. epi?
  Consider $g : X = A \to A \oplus B$, $g(a) = (a, 0)$. Then $f \circ g(a) = f(a, 0) = a \implies f \circ g$ onto.
  If $B \neq (0)$ then g is not onto $\implies$ f is not an essential epimorphism.

(2) $\Gamma :\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, k field, $\Lambda = k\Gamma$

$$\begin{tikzcd} & k \arrow[d, "1"] & & 0 \arrow[d] \\ A = \Lambda e_1 \arrow[r, squiggly] & k \arrow[d, "1"] & \supseteq & 0 \arrow[d] \\ & k & & 0 \\ & \text{\rotatebox{90}{=}} & & \text{\rotatebox{90}{=}} \\ & A & & B \end{tikzcd}$$

Let $f : A \to A/B$ be the natural epimorphism/projection. Let $g : X \to A$ and assume $g : X \to A$ is not onto.
<u>WTS:</u> $f \circ g$ not onto.

$$\text{Proper submodules of } A\text{:}\quad \begin{array}{c}0\\ \downarrow\\ k\\ \downarrow{\scriptstyle 1}\\ k\end{array} \supseteq \begin{array}{c}0\\ \downarrow\\ 0\\ \downarrow\\ k\end{array} \supseteq \begin{array}{c}0\\ \downarrow\\ 0\\ \downarrow\\ 0\end{array}$$

$$g \text{ not onto } \implies \operatorname{Im} g \subseteq \begin{array}{c}0\\ \downarrow\\ k\\ \downarrow{\scriptstyle 1}\\ k\end{array}$$

$$\implies \operatorname{Im} f\circ g \subseteq f\left(\begin{array}{c}0\\ \downarrow\\ k\\ \downarrow{\scriptstyle 1}\\ k\end{array}\right) = \begin{array}{c}0\\ \downarrow\\ k\\ \downarrow{\scriptstyle 1}\\ k\end{array} \Big/ \begin{array}{c}0\\ \downarrow\\ 0\\ \downarrow\\ k\end{array} \simeq \begin{array}{c}0\\ \downarrow\\ k\\ \downarrow\\ 0\end{array} \subsetneq A/B = \begin{array}{c}k\\ \downarrow{\scriptstyle 1}\\ k\\ \downarrow\\ 0\end{array}$$

$\implies f\circ g$ not onto $\implies f$ essential epimorphism.

(3) Λ left artinian, A finitely generated Λ-module
Claim: $P_A : A \to A/\mathfrak{r}A$ essential epimorphism.

Proof. Let $g : X \to A$, and assume that $f \circ g$ is onto

$$\begin{tikzcd} X \arrow[r, "P_X"] \arrow[d, "g"] & X/\mathfrak{r}X \arrow[d, "\overline{g}"] \\ A \arrow[r, "P_A"] & A/\mathfrak{r}A \end{tikzcd}$$

Know: $g : X \to A$ onto $\iff \overline{g} : X/\mathfrak{r}X \to A/\mathfrak{r}A$ is onto (Lemma 21)
$P_A \circ g = \overline{g} \circ P_X$ and $P_A \circ g$ onto $\implies \overline{g}$ onto $\implies g : X \to A$ onto
$\implies P_A$ essential epimorphism.

□

Exercise 6.8. $\left.\begin{array}{ll} f : A \to B & \text{ess. epi} \\ g : B \to C & \text{ess. epi}\end{array}\right\} \implies g\circ f : A \to C$ ess. epi

Proposition 22. *Λ left artinian, A, B finitely generated Λ-modules. Let $f : A \to B$ be onto. TFAE:*

(a) f is an essential epimorphism.
(b) $\operatorname{Ker} f \subseteq \mathfrak{r}A$, ($\mathfrak{r} = \operatorname{rad}\Lambda$)
(c) $\overline{f} : A/\mathfrak{r}A \to B/\mathfrak{r}B$ is an isomorphism.

Proof. $\underline{(a) \Rightarrow (b)}$: Assume f is an essential epimorphism. WTS: $\operatorname{Ker} f$ is small in A, i.e. $\overline{\operatorname{Ker} f \subseteq \mathfrak{r}A}$ (Prop 19 and Theorem 20)

Let $X \subseteq A$. Assume that $\operatorname{Ker} f + X = A$. Then the composition $X \overset{i}{\hookrightarrow} A \xrightarrow{f} B$ is onto (proof?), such that $X \overset{i}{\hookrightarrow} A$ is onto, since f is an essential epimorphism $\implies X = A \implies \operatorname{Ker} f$ is small in $A \implies \operatorname{Ker} f \subseteq \mathfrak{r}A$

$\underline{(b) \Rightarrow (c)}$: Assume that $\operatorname{Ker} f \subseteq \mathfrak{r}A$.

We have $\begin{array}{ccc} A/\operatorname{Ker} f & \overset{f'}{\underset{\sim}{\longrightarrow}} & \operatorname{Im} f = B \\ a + \operatorname{Ker} f & \longmapsto & f(a) \end{array}$ (f onto)

and therefor $\mathfrak{r}(A/\operatorname{Ker} f) = \mathfrak{r}A/\operatorname{Ker} f \overset{\simeq}{\longrightarrow}^{f'} \mathfrak{r}B$ and $(A/\operatorname{Ker} f)/(\mathfrak{r}A/\operatorname{Ker} f) \simeq B/\mathfrak{r}B$

$$\begin{tikzcd}
a+\mathfrak{r}A \arrow[r, mapsto] \arrow[dddd, mapsto, bend right] & f(a)+\mathfrak{r}B \\
A/\mathfrak{r}A \arrow[r, "\overline{f}"] & B/\mathfrak{r}B \\
\wr & \\
(A/\operatorname{Ker} f)/(\mathfrak{r}A/\operatorname{Ker} f) \arrow[uur, "\overline{f'}\sim"] & f(a)+\mathfrak{r}B \\
(a+\operatorname{Ker} f)+\mathfrak{r}A/\operatorname{Ker} f \arrow[ur, mapsto] &
\end{tikzcd}$$

$\overline{f}$ is the composition of two isomorphisms $\implies \overline{f}$ is an isomorphism.

<u>$(c) \Rightarrow (a)$:</u> Assume that $\overline{f} : A/\mathfrak{r}A \to B/\mathfrak{r}B$ is an isomorphism. Let $g : X \to A$

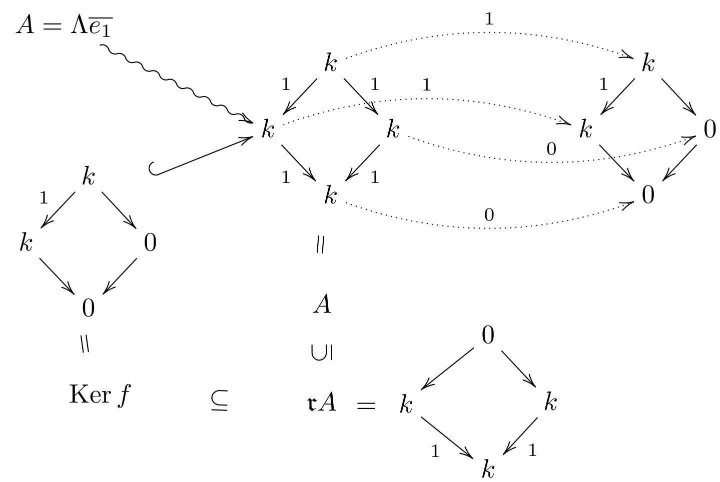
$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "P_X"] & A \arrow[r, "f"] \arrow[d, "P_A"] & B \arrow[d, "P_B"] \\
X/\mathfrak{r}X \arrow[r, "\overline{g}"] & A/\mathfrak{r}A \arrow[r, "\overline{f}", "\sim"'] & B/\mathfrak{r}B
\end{tikzcd}$$

Asuume $f \circ g$ is onto
$\implies \overline{f \circ g} = \overline{f} \circ \overline{g} : X/\mathfrak{r}X \to B/\mathfrak{r}B$ is onto
$\implies \overline{g} = \overline{f}^{-1}\overline{f}\overline{g} : X/\mathfrak{r}X \to A/\mathfrak{r}A$ is onto
Lemma 21 $\implies g$ is onto $\implies f$ is essential epimorphsim.

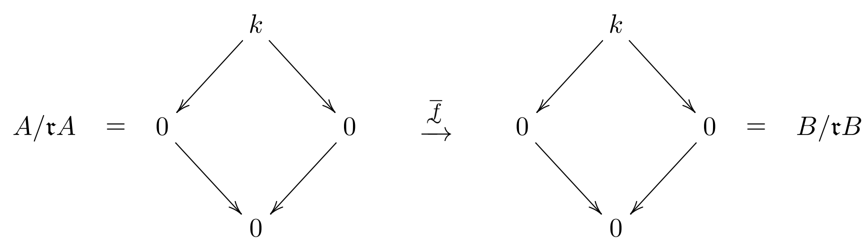
$\square$

Example 6.9. $\Gamma :$ $\begin{tikzcd}
 & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\
 & 4 &
\end{tikzcd}$ $\rho = \{\gamma\alpha - \delta\beta\}, k$ field, $\Lambda = k\Gamma/\langle\rho\rangle$

$B =$ $\begin{tikzcd}
 & k \arrow[dl, "1"'] \arrow[dr] & \\
k \arrow[dr] & & 0 \arrow[dl] \\
 & 0 &
\end{tikzcd}$ Find some $f : A \to B$ which is essential epimorphism.



$\Longrightarrow$ f is essential epimorphism, or



7. Projective Modules

Λ a ring, P a Λ-module

Definition 7.1. P is projective if for every epimorphism $g : B \to C$ and every Λ-homomorphism $f : P \to C$ there exists a homomorphism $h : P \to B$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

commutes.

Example 7.2. Λ is a projective Λ-module

$$\begin{tikzcd} & \Lambda \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & 1 \arrow[dd, mapsto, bend right] \\ B \arrow[r, "g"] & C \arrow[r] & 0 \\ b \arrow[rr, mapsto] & & f(1) \end{tikzcd}$$

$f(\lambda) = f(\lambda \cdot 1) = \lambda f(1)$. Choose $b \in B$ such that $g(b) = f(1)$, and define $h : \Lambda \to B$ by $h(\lambda) = \lambda b$

<u>Check:</u> h is Λ-homomorphism.
Then $gh(\lambda) = g(\lambda b) = \lambda g(b) = \lambda f(1) = f(\lambda \cdot 1) = f(\lambda), \forall \lambda \in \Lambda$
$\implies$ Λ projective.

Exercise 7.3. F free Λ-module $\implies$ F is projective ($F \simeq \bigoplus_{i \in I} \Lambda a_i$, ${}_\Lambda \Lambda a_i \simeq_\Lambda \Lambda$, $\forall i \in I$)

Proposition 23. *Λ a ring, P a projective Λ-module*
P projectiv $\iff$ $\exists$ free Λ-module F and a Λ-module Q such that $F \simeq P \oplus Q$

Proof. <u>$\Rightarrow$:</u> Assume that P is projective. Any module M is a factor of a free module F_M : Let $F_M = \bigoplus_{m \in M} \Lambda_m$, ${}_\Lambda \Lambda_m =_\Lambda \Lambda$ and define $\varphi : F_M \to M$ by $\varphi((\lambda_m)_{m \in M}) = \sum_{m \in M} \lambda_m m$

<u>Check:</u> φ Λ-homomorphism, φ onto, $m = \varphi((\lambda_x)_{x \in M})$, where $\lambda_x = \begin{cases} 0, & x \neq m \\ 1, & x = m \end{cases}$

Let $g : F \to P$ be onto with F free.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "h"'] \arrow[d, equal, "1_P"] & \text{projective} \arrow[l] \\
F \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

$$b \longmapsto f(1)$$

$\implies$ $\exists h : P \to F$ such that $gh = 1_P$

<u>Calim:</u> $F \simeq \operatorname{Ker} g \oplus \operatorname{Im} h$ (Excersice)

<u>$\Leftarrow$:</u> Assume that $F \overset{\varphi}{\xrightarrow{\sim}} P \oplus Q$ where F is a free Λ-module. Suppose that $g : B \to C$ is onto nd let $f : P \to C$.

$$(p, q) \longmapsto p$$

$$\begin{tikzcd}
F \arrow[r, "\sim", "\varphi"'] & P \oplus Q \arrow[r, "\pi"] \arrow[d, dashed, "\exists h'"'] & P \arrow[l, dashed, bend left, "\nu"] \arrow[d, "f"] & \\
 & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

$\nu(p) = (p, 0)$, $\pi\nu(p) = \pi(p, 0) = p$
F projective $\implies$ $\exists h' : F \to B$ such that

$$\begin{aligned}
gh' &= f\pi\varphi & &\mid \cdot\varphi^{-1} \\
gh'\varphi^{-1} &= f\pi & &\mid \cdot\nu \\
g(\underbrace{h'\varphi\nu}_{h}) &= f\phi\nu = f 1_P = f & &
\end{aligned}$$

$\implies$ P is projective. $\square$

Example 7.4. (Γ, ρ) a quiver with relations, k a field, $J^t \subseteq \langle \rho \rangle \subseteq J^2$, $\Gamma_0 = \{1, 2, \cdots, n\}$. $\Lambda = k\Gamma / \langle \rho \rangle$

<u>Have seen:</u> ${}_\Lambda\Lambda = \Lambda\overline{e_1} \oplus \Lambda\overline{e_2} \oplus \cdots \oplus \Lambda\overline{e_n}$
$\implies \Lambda\overline{e_i}$ projective Λ-modules

Definition 7.5. Let $f : P \to M$ be a Λ-homomorphism. Then $f : P \to M$ is a <u>projective cover of M</u> if P is projectiv and f is an essential epimorphism.

Example 7.6. $\Gamma :$ $\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$, $\rho = \{\gamma\alpha - \delta\beta\}$, $\Lambda = k\Gamma/\langle\rho\rangle$, $M =$ $\begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr] & \\ k \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd}$

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r] & \Lambda\overline{e_1} \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

$$\operatorname{Ker} f = \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd} \qquad \Lambda\overline{e_1} = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\ k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd} \qquad M = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr] & \\ k \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd}$$

We have $\operatorname{Ker} f \subseteq \mathfrak{r}(\Lambda\overline{e_1})$
$\implies f$ is essential epimorphism
$\implies f$ is projective cover.

Theorem 24. Λ *left artinian,* A *finitely generated*

(a) There exists a projective cover $f : P \to A$ (P finitely generated)
(b) Let $f_1 : P_1 \to A$ and $f_2 : P_2 \to A$ be two projective covers of A. Then there exists an isomorphism $g : P_1 \to P_2$ such that $f_1 = f_2 g$

$$\begin{tikzcd} P_1 \arrow[dr, "f_1"] \arrow[dd, "g"', "\wr"] & \\ & A \\ P_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

Proof. (a) A finitely generated $\implies \exists$ onto $f : \Lambda^n \to A$, $n < \infty$.
Choose P projective, $f : P \to A$ an onto Λ-homomorphism with $l(P)$ minimal. <u>WTS:</u> f projective cover. Consider

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r, hook] & P \arrow[r, "f"] \arrow[d, "\pi_P"] & A \arrow[r] \arrow[d, "\pi_A"] & 0 \\ & & P/\mathfrak{r}P \arrow[r, "\overline{f}"] & A/\mathfrak{r}A \arrow[r] & 0 \end{tikzcd}$$

Assume that $\operatorname{Ker} f \not\subseteq \mathfrak{r}P$. Then $\pi_P(\operatorname{Ker} f) \neq (0)$ in $P/\mathfrak{r}P$, which is semisimple. We know that $P/\mathfrak{r}P = \pi_P(\operatorname{Ker} f) \oplus X$ for some $X \subseteq P/\mathfrak{r}P$.
Define $e : P/\mathfrak{r}P \to P/\mathfrak{r}P$ by letting $e(v, w) = (0, w)$ for $(v, w) \in P/\mathfrak{r}P = \pi_P(\operatorname{Ker} f) \oplus X$.

$$\begin{tikzcd} P \arrow[r, dashed, "e'"] \arrow[d, "\pi_P"] & P \arrow[d, "\pi_P"] \\ P/\mathfrak{r}P \arrow[r, "e"] & P/\mathfrak{r}P \arrow[d] \\ & 0 \end{tikzcd}$$

Then $\pi_P e'e' = ee\pi_P = e\pi_P = \pi_P e' \implies \pi_P(e' - e'e') = 0$. Hence, if $x = e'(1_P - e') = (1_P - e')e'$ then $\operatorname{Im} x \subseteq \mathfrak{r}P$. Since $\operatorname{Im} x^2 \subseteq \mathfrak{r}^2 P$ and so on, and $\mathfrak{r}^t = (0)$ for some $t \geq 1$, we have that $x^t = 0$. Let $a = e'$ and $b = 1_P - e'$. Then

$$a^t b^t = b^t a^t = x^t = 0$$

and

$$1_P = (a+b)^{2t} = \underbrace{a^{2t} + r_1 a^{2t-2} b + \cdots r_t a^t b^t}_{\tilde{e}} + \underbrace{r_{t+1} a^{t-1} b^{t+1} + \cdots + r_{2t-1} a b^{2t-1} + b^{2t}}_{\tilde{e}'}$$

It follows that $\tilde{e} = \tilde{e} \cdot 1_P = \tilde{e}(\tilde{e} + \tilde{e}') = \tilde{e}^2 + \tilde{e}\tilde{e}' = \tilde{e}^2 + 0 = \tilde{e}^2$
$\implies$ $\tilde{e}$ is aan idempotent.
<u>Check:</u> $\overline{\tilde{e}} = e^{2t} = e$ and $\overline{f}(P/\mathfrak{r}P) = \overline{f}(X) = \overline{f}(\operatorname{Im} e)$.

Then we have

$$\begin{tikzcd} P \arrow[r, "\tilde{e}"] \arrow[d, "\pi_P"'] & P \arrow[r, "f"] \arrow[d, "\pi_P"'] & A \arrow[d, "\pi_A"'] \\ P/\mathfrak{r}P \arrow[r, "e"] & P/\mathfrak{r}P \arrow[r, "\overline{f}"] & A/\mathfrak{r}A \end{tikzcd}$$

$$\underbrace{\overline{f} e \pi_P}_{\text{onto}} \overset{\textstyle =}{\implies} \underbrace{\pi_A f \tilde{e}}_{\text{onto}}$$

<u>Seen:</u> π_A essential epimorphism $\implies f\tilde{e}$ onto $\implies f\,|_{\operatorname{Im}\tilde{e}} \colon \operatorname{Im}\tilde{e} \to A$ onto.

$$\begin{array}{lccccc} \text{We have} & P & = & \operatorname{Im}\tilde{e} & \oplus & \operatorname{Ker}\tilde{e} \\ & & & \uparrow & & \parallel \\ \text{Check:} & & & \text{projective} & & \operatorname{Im}(1_P - \tilde{e}) \end{array}$$

$\overline{(1_P - \tilde{e})} = 1 - e \neq 0 \implies \operatorname{Im}(1_P \tilde{e}) \neq (0)$.
$\implies l(\operatorname{Im}\tilde{e}) < l(P)$, Contradiction!
$\implies \operatorname{Ker} f \subseteq \mathfrak{r}P$ and f is an essential epimorphsim.
f is projective cover.

(b) Let $f_i : P_i \to A$ be a projective cover of A for $i = 1, 2$.

$$\begin{tikzcd}
P_1 \arrow[d, "\exists g"'] \arrow[dr, "f_1"] & & 0 \\
P_2 \arrow[r, "f_2"] \arrow[d, "\exists h"'] & A \arrow[ur] \arrow[r] \arrow[dr] & 0 \\
P_1 \arrow[ur, "f_1"'] & & 0
\end{tikzcd}$$

P_1 projective and f epimorphic $\implies \exists g : P_1 \to P_2$ such that $f_1 = f_2 g$.
f_2 essential epimorphism and f_1 epimorphic $\implies g$ epimorphic.
Similarly h is epimorphic.
$\implies l(P_2) \leq l(P_1) \leq l(P_1) \implies l(P_1) = l(P_2) \implies g$ isomorphism.

□

Proposition 25. *Λ left artinian, $f : P \to A$ an epimorphism, A finitely generated, P projective Λ-module.*

(a) $f : P \to A$ projective cover $\iff \overline{f} : P/\mathfrak{r}P \to A/\mathfrak{r}A$ isomorphic.
(b) $\{f_i : P_i \to A_i\}_{i=1}^m$, f_i onto, P_i projective
The map $f : P_1 \oplus P_2 \oplus \cdots \oplus P_m \to A_1 \oplus A_2 \oplus \cdots \oplus A_m$ defined by $(p_1, p_2, \cdots, p_m) \mapsto (f_1(p_1), f_2(p_2), \cdots, f_m(p_m))$ is a projectiv cover $\iff$ each $f_i : P_i \to A$ is a projectiv cover.

Proof.
(a) Prop **??**
(b) <u>Exercise:</u> Use that $\mathfrak{r}(P_1 \oplus \cdots \oplus P_m) = \mathfrak{r}P_1 \oplus \cdots \oplus \mathfrak{r}P_m$, $\mathfrak{r} = \operatorname{rad}\Lambda$ and (a)

□

Proposition 26. *Λ left artinian, finitely generated modules.*

(a) P projective module then $P \to P/\mathfrak{r}P$ is projective cover.
(b) P, Q projective modules, $P \simeq Q \iff P/\mathfrak{r}P \simeq Q/\mathfrak{r}Q$
(c) P projective module, P indecomposable $\iff P/\mathfrak{r}P$ simple.
(d) Assume that $P = \bigoplus_{i=1}^n P_i \simeq \bigoplus_{j=1}^m Q_j$, with P projective and P_i and Q_j indecomposable.
$\implies n = m$ and $\exists$ permutation π of $\{1, 2, \cdots, n\}$ such that $P_i \simeq Q_{\pi(i)}$, for $i = 1, 2, \cdots, n$.

Proof.
(a) <u>We have seen:</u> $A \to A/\mathfrak{r}A$ is essential epimorphism for finitely generated A.
P projective $\implies P \to P/\mathfrak{r}P$ projective cover.
(b) <u>$\Rightarrow$:</u> Clear.
<u>$\Leftarrow$:</u> Assume that $P/\mathfrak{r}P \simeq Q/\mathfrak{r}Q$.
$(a) \implies P \to P/\mathfrak{r}P, Q/\mathfrak{r}Q$ are projective covers. Assumption plus projective cover unique up to isomorphism (Theorem 24 (b)) $\implies P \simeq Q$.

(c) $\underline{\Leftarrow:}$ Assume that $P = P_1 \oplus P_2$ with $P_i \neq (0)$. Since $\mathfrak{r}P = \mathfrak{r}P_1 \oplus \mathfrak{r}P_2$ we have that

$$P/\mathfrak{r}P \;=\; P_1 \oplus P_2/\mathfrak{r}P_1 \oplus \mathfrak{r}P_2 \;\simeq\; \underset{(0)}{\underset{\nparallel}{P_1/\mathfrak{r}P_1}} \;\oplus\; \underset{(0)}{\underset{\nparallel}{P_2/\mathfrak{r}P_2}}$$

Nakayama Lemma

$\implies P/\mathfrak{r}P$ not simple

$\underline{\Rightarrow:}$ Assume that $P/\mathfrak{r}P$ is not simple. $P/\mathfrak{r}P$ is a semisimple Λ-module $\implies P/\mathfrak{r}P = U_1 \oplus U_2$, where $U_i \neq (0)$, $i = 1, 2$.
Let P_i be a projective cover of U_i for $i = 1, 2$.
(a) gives that $P \to P/\mathfrak{r}P$ is a projective cover.
<u>We know:</u> $P_1 \oplus P_2 \to U_1 \oplus U_2 = P/\mathfrak{r}P$ is a projective cover.
Uniqueness (Theorem 24 (b)) $\implies P \simeq P_1 \oplus P_2$ with $P_i \neq (0)$, $i = 1, 2$.
$\implies P$ decomposes.

d <u>Exercise:</u>
<u>Note:</u>
$$\begin{array}{ccc} P_i/\mathfrak{r}P_i \hookrightarrow & P_1/\mathfrak{r}P_1 \oplus \cdots P_n/\mathfrak{r}P_n & \\ & \wr\| & \\ & Q_1/\mathfrak{r}Q_1 \oplus \cdots Q_n/\mathfrak{r}Q_n & \twoheadrightarrow Q_j/\mathfrak{r}Q_j \end{array}$$
$\implies \exists j$ such that $Q_j/\mathfrak{r}Q_j \simeq P_i/\mathfrak{r}P_i \implies P_i \simeq Q_j$

$\square$

Recall 7.7. Λ left artinian.
$\implies \Lambda/\mathfrak{r} = S_1 \oplus \cdots \oplus S_n$, S_i simple. Furthermore if T is a simple Λ-module then $\exists j$ such that $T \simeq S_j$, i.e $\{S_i\}_{i=1}^n$ has representatives from all isomorphism classes of simple Λ-modules.

Let $P_i \overset{f_i}{\twoheadrightarrow} S_i$ be a projective cover of S_i.
Prop 25 (a) $\implies P_i/\mathfrak{r}P_i \simeq S_i$, since $\mathfrak{r}S_i = (0)$.
Prop 25 (c) $\implies P_i$ indecomposable.

Let P be an indecomposable projective Λ-module. Then $P \to P/\mathfrak{r}P$ is a projective cover and $P/\mathfrak{r}P$ is simple. Hence $\exists j$ such that $P/\mathfrak{r}P \simeq S_j$.

$$P \to P/\mathfrak{r}P \simeq S_j \leftarrow P_j \text{ projective cover}$$

Uniqueness $\implies P \simeq P_j$.

Corollary 27. *Λ left artinian. The only indecomposable projective Λ-modules are $P_1, P_2, \cdots, P_n$ up to isomorphsim.*

Example 7.8. Γ :
$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 2 \arrow[dr, "\gamma"'] & & 3 \arrow[dl, "\delta"] \\ & 4 & \end{tikzcd}$$
, $\rho = \{\gamma\alpha - \delta\beta\}$, k field $\Lambda = k\Gamma/\langle\rho\rangle$.

<u>We have seen:</u>
$$\Lambda/\mathfrak{r} \;=\; \underset{S_1}{\underset{\|}{k\bar{e}_1}} \;\oplus\; \underset{S_2}{\underset{\|}{k\bar{e}_2}} \;\oplus\; \underset{S_3}{\underset{\|}{k\bar{e}_3}} \;\oplus\; \underset{S_4}{\underset{\|}{k\bar{e}_4}}$$

Projective covers:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & k \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[rrr, dotted, bend left, "1"] & & & k \arrow[dl] \arrow[dr] & & \\
\Lambda\bar{e}_1 : & k \arrow[dr, "1"'] & & k \arrow[dl, "1"] & 0 \arrow[dr] & & 0 \arrow[dl] & = S_1 \\
 & & k & & & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & & 0 \arrow[dl] \arrow[dr] & & & 0 \arrow[dl] \arrow[dr] & & \\
\Lambda\bar{e}_2 : & k \arrow[dr, "1"'] \arrow[rrr, dotted, bend right, "1"] & & 0 \arrow[dl] & k \arrow[dr] & & 0 \arrow[dl] & = S_2 \\
 & & k & & & 0 & &
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & 0 \arrow[dl] \arrow[dr] & & & 0 \arrow[dl] \arrow[dr] & & \\
\Lambda\bar{e}_3 : & 0 \arrow[dr] & & k \arrow[dl, "1"] \arrow[rrr, dotted, bend right, "1"] & 0 \arrow[dr] & & k \arrow[dl] & = S_3 \\
 & & k & & & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & & 0 \arrow[dl] \arrow[dr] & & & & 0 \arrow[dl] \arrow[dr] & & \\
\Lambda\bar{e}_4 : & 0 \arrow[dr] & & 0 \arrow[dl] & = & 0 \arrow[dr] & & 0 \arrow[dl] & = S_4 \\
 & & k & & & & k & &
\end{tikzcd}$$

Lemma 28. Λ *left artinian, M finitely generated.*
Let $P \xrightarrow{f'} M/\mathfrak{r}M$ *be a projective cover. Then a homomorphism $f : P \to M$ such that*

$$\begin{tikzcd}
 & P \arrow[dl, "f"'] \arrow[d, "f'"] \\
M \arrow[r, "\pi_M"] & M/\mathfrak{r}M
\end{tikzcd}$$

commutes, is a projective cover of M.

Proof.

$$\left.\begin{array}{ll} f' = \pi_M \cdot f & \text{epi.} \\ M \xrightarrow{\pi_M} M/\mathfrak{r}M & \text{ess. epi.} \end{array}\right\} \implies f \text{ epimorphism}$$

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, equal] & M \arrow[d, "\pi_M"] \\
P \arrow[r, "f'"] \arrow[d, "\pi_P"] & M/\mathfrak{r}M \arrow[d, equal] \\
P/\mathfrak{r}P \arrow[r, "\overline{f}"] & M/\mathfrak{r}M
\end{tikzcd}$$

Know: $\overline{f'} : P/\mathfrak{r}P \xrightarrow{\sim} M/\mathfrak{r}M$ isomorphism.

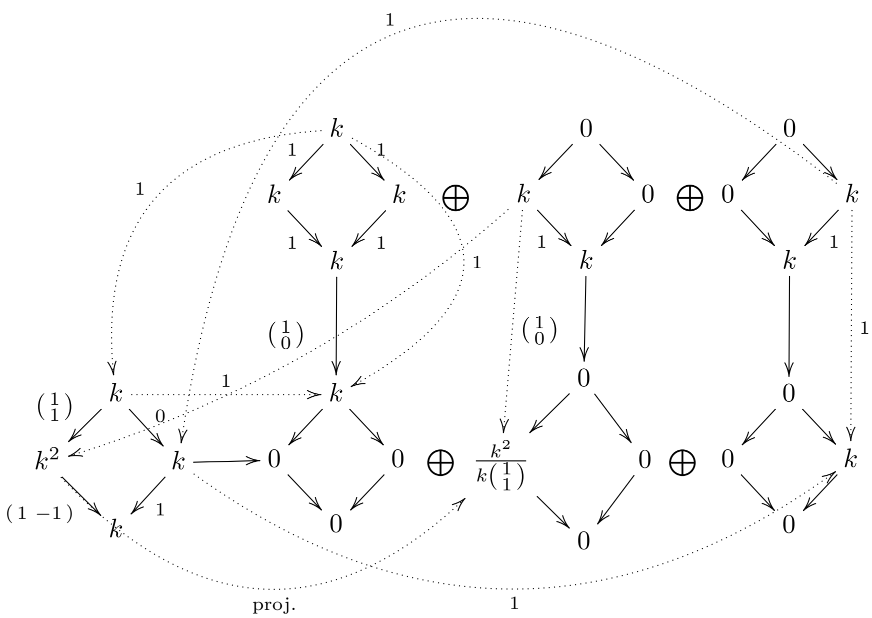
We have: $\overline{f}\pi_P = \pi_M f = f' = \overline{f'}\pi_P$ $\qquad \pi_P$ epimorphic

$\implies \overline{f} = \overline{f'} \leftarrow$ isomorphism
$\implies \overline{f}$ isomorphism $\implies f : P \to M$ projective cover. ∎

Example 7.9.

$$M: \quad \begin{tikzcd} & k \arrow[dl, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] \arrow[dr, "0"] & \\ k^2 \arrow[dr, "(1\ -1)"'] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd} \qquad \mathfrak{r}M: \quad \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ k\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \arrow[dr] & & 0 \arrow[dl] \\ & k & \end{tikzcd}$$

$$M/\mathfrak{r}M: \quad \begin{tikzcd} & k \arrow[dl, "0"'] \arrow[dr] & \\ k^2/k\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \arrow[dr] & & 0 \arrow[dl] \\ & 0 & \end{tikzcd} \qquad \simeq S_1 \oplus S_2 \oplus S_3$$

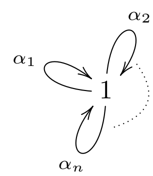


Definition 7.10. Λ is a <u>local ring</u> if the non-invertible elements in Λ is an ideal.

Example 7.11.

(1) $\mathbb{Z}$ is not a local ring. Invertible elements: $\{-1, 1\}$, but $3 + (-2) = 1 \implies$ non-invertible elements is not an ideal.
(2) k fiel $\implies$ k local
(3) $\Lambda_n = k[x]/\langle x^n \rangle$, k a field.
Non-invertibel elements: $\langle x \rangle / \langle x^n \rangle$.
Invertible elements: $\{a + f(x)x + \langle x^n \rangle \mid a \in k \setminus \{0\}, f(x) \in k[x]\}$.
$\implies$ Λ is a local ring.

(4)

Γ :  , ρ relation such that $J^t \subseteq \langle \rho \rangle \subseteq J^2$

$\Lambda = k\Gamma/\langle \rho \rangle$ - finitedimensional k-algebra, k field.
Non-invertible elements: $J/\langle \rho \rangle = \mathfrak{r}$ - ideal
Invertible elements: $\{ae_1 + r \mid a \in k \setminus \{0\}, r \in \mathfrak{r}\}$
$\implies \Lambda$ a local ring.

Proposition 29. *Λ a local ring $\implies$ 0 and 1 are the only idempotents.*

Proof. Let e be an idempotent in Λ. Suppose that e is an invertible element, i.e. $\exists f \in \Lambda$ such that $ef = fe = 1$.
$\implies e = e \cdot 1 = e(ef) = e^2 f = ef = 1$.
Suppose that $e \neq 0, 1$, i.e. e and $1 - e$ are both not invertible. Then

$$1 = e + (1 - e)$$

not invertible

Λ local $\implies$ 1 not invertible. Contradiction!
$\implies 0, 1$ the only idempotents in Λ. □

Note 7.12.
(1) $\Lambda = k\Gamma/\langle \rho \rangle$, k a field $J^t \subseteq \langle \rho \rangle \subseteq J^2$, $|\Gamma_0| \geq 2 \implies \Lambda$ not local (Why?)
(2) Λ a ring, $M \neq (0)$ a Λ-module.
We have seen: $\mathrm{End}_\Lambda(M)$ contains an idempotenet $\neq 0, 1 \iff M$ decomposes.

Corollary 30. *M Λ-module, $\mathrm{End}_\Lambda(M)$ a local ring $\implies M$ indecomposable.*

Proposition 31. *Λ left artinian, P finitely generated projective Λ-module. TFAE:*

(a) P indecomposable
(b) $\mathrm{End}_\Lambda(P)$ local
(c) $\mathfrak{r}P$ is the only maxiaml submodule of P.
(d) $P/\mathfrak{r}P$ is simple.

Proof.
(a) $\iff$ (d) : Prop 26 (c).
(d) $\Rightarrow$ (c) : $P/\mathfrak{r}P$ simple $\implies \mathfrak{r}P$ maximal submodule of P.

$$\underline{\text{We have:}}\ \mathfrak{r}P = \operatorname{rad} P = \bigcap_{\substack{M \text{ max.} \\ \text{submod. of } P}} M$$

$\implies \mathfrak{r}P \subseteq M, \forall M \subseteq P$ maximal submodule.
$\implies \mathfrak{r}P$ is the only maximal submodule of P.

$\underline{(c) \Rightarrow (b):}$ Let $f : P \to P$. Then

$$\begin{aligned} f \text{ invertible} &\overset{\text{def}}{\iff} f \text{ isomorphism} \\ &\iff f \text{ is an epimorphsm } (l(P) < \infty) \quad \text{Hence, \{non-invertable} \\ &\overset{\mathfrak{r}P \text{ max}}{\iff} \operatorname{Im} f \not\subseteq \mathfrak{r}P \end{aligned}$$

elements in $\operatorname{End}_\Lambda(P)\} = \{f \in End_\Lambda(P) \mid \operatorname{Im} f \subseteq \mathfrak{r}P\} = I$

$\underline{I \text{ ideal:}}$ $f_1, f_2 \in I$, $p \in P$

$$(f_1 - f_2)(p) = \underbrace{f_1(p)}_{\in} - \underbrace{f_2(p)}_{\ni} \in \mathfrak{r}P \Rightarrow f_1 - f_2 \in I$$

$$\mathfrak{r}P$$

$f_1, f_2 \in \operatorname{End}_\Lambda(P)$:

$$\begin{array}{ccccc} \operatorname{Im} f_1 f_2 = & f_1(\operatorname{Im} f_2) & \subseteq & \mathfrak{r}P, & \text{if } f_1 \in I \\ & \cap\!\!\vert & & & \\ & f_1(\mathfrak{r}P) & & \text{if } f_2 \in I & \\ & \Vert & & & \\ & \mathfrak{r}f_1(P) & \subseteq & \mathfrak{r}P & \end{array}$$

$\implies f f_1, f_1 f \in I$, when $f_1 \in I$ and $f \in \operatorname{End}_\Lambda(P)$.

$\implies I$ is an ideal $\implies \operatorname{End}_\Lambda(P)$ a local ring.

$\underline{(b) \Rightarrow (a):}$ Corollary 30

□

Example 7.13.

(1) $\mathbb{Z}$ projective $\mathbb{Z}$-module, $\mathbb{Z}$ not artinian, $End_{\mathbb{Z}}(\mathbb{Z}) \simeq \mathbb{Z}$ not local, but $\mathbb{Z}$ indecomposable.

(2) $\Lambda = k\Gamma/\langle\rho\rangle$, k field, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, Λ left artinian.
<u>We have:</u> $\Lambda\overline{e}_i$ indecomposable projective Λ-modules.
Prop 31 $\implies \operatorname{End}_\Lambda(\Lambda\overline{e}_i) \simeq \overline{e}_i\Lambda\overline{e}_i$ is a local ring.

Corollary 32. Λ *left artinian. TFAE:*

(a) Λ *local*
(b) $\mathfrak{r} = \operatorname{rad}\Lambda$ *is a maximal left ideal*
(c) $\Lambda/\mathfrak{r}$ *is simple* Λ*-module*

Proof. Follows from Prop 31, noting that $\Lambda \simeq \operatorname{End}_\Lambda(\Lambda)^{op}$ □

Proposition 33. Λ *left artinian*

(a) $1 = e_1 + e_2 + \cdots + e_n$ *- a sum of primitive orthogonal idempotents.*
orthogonal: $e_i e_j = 0$, $\forall i \neq j$
primitive: $e_i \neq 0$ *is not a sum of non-zero idempotents*
(b) Let $e_1, e_2, \cdots, e_m$ *be idempotents in* Λ*, and let* $e = e_1 + e_2 + \cdots + e_m$*.*
If $e_1, e_2, \cdots, e_m$ *are orthogonal then* $\Lambda e = \Lambda e_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_m$*.*
(c) Let $e \neq 0$ *be an idempotent. Then* Λe *indecomposable* $\iff$ *e is primitive.*

Proof. (a) $\Lambda/\mathfrak{r}$ semisimple
$\implies \Lambda/\mathfrak{r} = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, S_i simple.
Let $P(S_i) \to S_i$ be the projective cover of S_i. Then $P(S_i)$ is indecomposable by Prop 26 (c). Then by Prop 25 (b): $\bigoplus_{i=1}^n P(S_i) \to \Lambda/\mathfrak{r}$ is a projective cover. Also $\Lambda \to \Lambda/\mathfrak{r}$ is a projective cover.

Uniqueness $\implies \Lambda \simeq \bigoplus_{i=1}^n P(S_i)$
$\implies \Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$, P_i indecomposable.
Then $1 = e_1 + e_2 + \cdots + e_n$ with $e_i \in P_i$

$$\implies \underset{\substack{\Cap \\ P_i}}{e_i} = e_i \cdot 1 = \underset{\substack{\Cap \\ P_1}}{e_ie_1} + \underset{\substack{\Cap \\ P_2}}{e_ie_2} + \cdots + \underset{\substack{\Cap \\ P_i}}{e_ie_i} + \cdots + \underset{\substack{\Cap \\ P_n}}{e_ie_n}$$

Direct sum $\implies e_ie_j = \begin{cases} 0, & \text{if } i \neq j \\ e_i, & \text{if } i = j \end{cases}$
$\implies \{e_i\}$ are orthogonal idempotents.

<u>Claim:</u> $P_i = \Lambda e_i$
$e_i \in P_i \implies \Lambda e_i \subseteq P_i$

$$x \in P_i = x = x \cdot 1 = \underset{\substack{\Cap \\ P_1}}{xe_1} + \underset{\substack{\Cap \\ P_2}}{xe_2} + \cdots + \underset{\substack{\Cap \\ P_i}}{xe_i} + \cdots + \underset{\substack{\Cap \\ P_n}}{xe_n}$$

Direct sum $\implies x = xe_i \in \Lambda e_i \implies P_i = \Lambda e_i$

<u>e_i primitive:</u> Let $e_i = f_1 + f_2$ with $f_i \neq 0$ fro $i = 1, 2$, and f_1 and f_2 are orthogonal idempotents. Using (b):

$$\Lambda e_i = \underset{\substack{\not\simeq \\ (0)}}{\Lambda f_1} \oplus \underset{\substack{\not\simeq \\ (0)}}{\Lambda f_2}$$

$\implies P_i$ not indecomposable. Contradiction!
$\implies e_i$ primitive for $i = 1, 2, \cdots, n$.

(b) and (c): Exercises.

□

Proposition 34. *Λ left artinian, P finitely generated projective Λ-module. Then*

$$P \simeq \bigoplus_{i=1}^n P_i$$

with P_i indecomposable and decomposition is unique up to isomorphism an ordering.

Proof. $P/\mathfrak{r}P = \bigoplus_{i=1}^n S_i$, S_i simple Λ-modules.
$P \to \mathfrak{r}P$ projective cover and $\bigoplus_{i=1}^n P(S_i) \to \bigoplus_{i=1}^n S_i = P/\mathfrak{r}P$ is a projective cover.

$$\text{Uniqueness} \implies P \simeq \bigoplus_{i=1}^n \underbrace{P(S_i)}_{indec}$$

□

8. Krull-Remak-Schmidth theorem

Lemma 35. *(Fitting Lemma)*
Λ a ring, M a Λ-module with $l(M) < \infty$, $\varphi \in \operatorname{End}_\Lambda(M)$. Then $\exists n \geq 1$ such that

$$M = \operatorname{Im} \varphi^n \oplus \operatorname{Ker} \varphi^n$$

Proof. $l(M) < \infty \implies M$ artinian and noetherian.

$\implies \left\{\begin{matrix} \operatorname{Im}\varphi & \supseteq & \operatorname{Im}\varphi^2 & \supseteq & \cdots \\ \operatorname{Ker}\varphi & \subseteq & \operatorname{Ker}\varphi^2 & \subseteq & \cdots \end{matrix}\right\}$ become stationary.

$\implies \exists n$ such that $\begin{matrix} \operatorname{Im}\varphi^n & = & \operatorname{Im}\varphi^{n+1} & = & \cdots \\ \operatorname{Ker}\varphi^n & = & \operatorname{Ker}\varphi^{n+1} & = & \cdots \end{matrix}$

$\implies l(\operatorname{Im}\varphi^n) = l(\operatorname{Im}\varphi^{2n})$, $\varphi^n : \operatorname{Im}\varphi^n \to \varphi^{2n}$ is surjective.

$\implies \varphi^n : \operatorname{Im}\varphi^n \underset{\psi}{\rightleftarrows} \operatorname{Im}\varphi^{2n}$ is an isomorphism.

Let $\psi : \operatorname{Im}\varphi^{2n} \to \operatorname{Im}\varphi$ be an inverse of φ^n.

(1) We have: $\operatorname{Im}\varphi^n, \operatorname{Ker}\varphi^n \subseteq M$

(2) $\underline{M = \operatorname{Im}\varphi^n + \operatorname{Ker}\varphi^n :}$ Let $m \in M$. Then

$$m = \underbrace{\psi\varphi^n(m)}_{\in \operatorname{Im}\varphi^n} + \underbrace{m - \psi\varphi^n(m)}_{\in \operatorname{Ker}\varphi^n}$$

Since $\varphi^n(m - \psi\varphi^n(m)) = \varphi^n(m) - \underbrace{\varphi^n\psi}_{1_{\operatorname{Im}\varphi^n}} \varphi^n(m) = 0$

(3) $\underline{\operatorname{Im}\varphi^n \cap \operatorname{Ker}\varphi^n = (0) :}$ Let $m \in \operatorname{Im}\varphi^n \cap \operatorname{Ker}\varphi^n$. Then $m = \varphi^n(m')$ for some $m' \in M$.
$m \in \operatorname{Ker}\varphi^n \implies 0 = \varphi^n(m) = \varphi^{2n}(m')$
$\implies m' \operatorname{Ker}\varphi^{2n} = \operatorname{Ker}\varphi^n$
$\implies m = \varphi^n(m') = 0$

$\square$

Theorem 36. Λ *left arinian,* M *finitely generated. Then*

$$M \text{ indecomposable} \iff \operatorname{End}_\Lambda(M) \text{ local}$$

Proof. $\underline{\Rightarrow:}$ Have seen (true in general).
$\underline{\Leftarrow:}$ Assume that M is indecomposable. Let $\varphi \in \operatorname{End}_\Lambda(M)$ be a non-invertible element. Then $l(\operatorname{Im}\varphi) < l(M)$
$\implies \forall \psi \in \operatorname{End}_\Lambda(M)$ the composition $\psi\varphi$ is not invertible ($l(\operatorname{Im}\psi\varphi) \leq l(\operatorname{Im}\varphi)$).
Fitting Lemma $\implies M \simeq \operatorname{Im}(\psi\varphi)^n \oplus \operatorname{Ker}(\psi\varphi)^n$
M indecomposable $\implies \operatorname{Im}(\psi\varphi)^n = (0)$ and $\operatorname{Ker}(\psi\varphi)^n = M$
or $\operatorname{Im}(\psi\varphi)^n = M$ and $\operatorname{Ker}(\psi\varphi)^n = (0)$
We have that $\operatorname{Im}(\psi\varphi)^n \subsetneq M$, so $\operatorname{Im}(\psi\varphi)^n = (0)$ and $\psi\varphi$ is nilpotent.
$\implies 1_M - \psi\varphi$ invertible in $\operatorname{End}_\Lambda(M)$ for all $\psi \in \operatorname{End}_\Lambda(M)$.
$\implies \varphi \in \operatorname{rad}\operatorname{End}_\Lambda(M) \subseteq \{$non-invertible elements of $\operatorname{End}_\Lambda(M)\}$
$\implies \operatorname{End}_\Lambda(M)$ is local. $\square$

Theorem 37. *(Krull-Remak-Schmidt theorem)*
Λ *left artinian,* M *finitely generated* Λ*-module.*

(a) M *can be written as a finite direct sum of indecomposable modules, i.e.*

$$M = \bigoplus_{i=1}^{n} M_i \text{ with } M_i \text{ indecomposable}$$

(b) The composition of M *into idecomposable modules is unique up to isomorphsim and ordering.*

Proof. (a) Induction on $l(M)$:
If $l(M) = 1$, then M is simple and clearly indecomposable. Claim in (a) trivially true.
Assume that (a) is true for all Λ-modules X with $l(X) < n$. Suppose $l(M) = n$. If M is indecomposable then we are done. If M decomposes say $M = M_1 \oplus M_2$ $(M_i \neq (0))$ then $l(M_i) < l(M)$ and we are done by induction.

(b) Assume that

$$M = \bigoplus_{i=1}^{n} M_i = \bigoplus_{j=1}^{m} N_j$$

with M_i and N_j indecomposable.

(a) If $l(M) = 1$ then the claim is true as M is simple and therefor indecomposable.

(b) Assume true for all modules X with $l(X) < n$. Let $l(M) = n$. Denote by φ_{sr} and ψ_{rs} the composition

$$M_r \hookrightarrow \bigoplus_{i=1}^{n} M_i = \bigoplus_{j=1}^{m} N_j \twoheadrightarrow N_s$$

and

$$N_s \hookrightarrow \bigoplus_{j=1}^{m} N_j = \bigoplus_{i=1}^{n} M_i \twoheadrightarrow M_r$$

respectively. Then

$$\sum_{s=1}^{m} \psi_{is}\varphi_{si} = 1_{M_i}$$

Since $\mathrm{End}_\Lambda(M_i)$ is local $\exists j$ such that $\psi_{ij}\varphi_{ji}$ is an isomorphism. If $\varphi_{ji}\psi_{ij} : N_j \to N_j$ is in $\mathrm{rad}\,\mathrm{End}_\Lambda(N_j)$, then it follows from Fitting Lemma that $\varphi_{ji}\psi_{ij}$ is nilpotent, say $(\varphi_{ji}\psi_{ij})^t = 0$
$\Longrightarrow$ $(\psi_{ij}\varphi_{ji})^{t+1} = 0$ Contradiction!
$\Longrightarrow$ $\varphi_{ji}\psi_{ij}$ is an isomorphsim
$\Longrightarrow$ φ_{ji} and ψ_{ij} are isomorphisms.
We have that

$$\begin{tikzcd}
M \arrow[r, phantom, "="] & \bigoplus_{r=1}^{n} M_r \arrow[r, "1_M = (\varphi_{ji})"] \arrow[d, phantom, "\parallel"] & \bigoplus_{s=1}^{n} N_s \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & M \\
& M_i \oplus \hat{M}_i \arrow[r, "\left(\begin{smallmatrix} \varphi_{ji} & a \\ b & c \end{smallmatrix}\right)"] & N_j \oplus \hat{N}_j \arrow[d, "\wr"', "\left(\begin{smallmatrix} 1_{N_j} & 0 \\ -b\varphi_{ji}^{-1} & 1_{\hat{N}_j} \end{smallmatrix}\right)"] & \\
& M_i \oplus \hat{M}_i \arrow[u, "\wr", "\left(\begin{smallmatrix} 1_{M_j} & -\varphi_{ji}^{-1}a \\ 0 & 1_{\hat{M}_j} \end{smallmatrix}\right)"'] \arrow[r, "A"] & N_j \oplus \hat{N}_j &
\end{tikzcd}$$

$a : \hat{M}_i \to N_j$
$b : M_i \to \hat{N}_j$
$c : \hat{M}_i \to \hat{N}_j$

Check:
is 1-1 $\longrightarrow$
$\Longrightarrow$ is onto

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 0 \\ -b\varphi_{ji}^{-1} & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \varphi_{ji} & a \\ b & c \end{pmatrix} \begin{pmatrix} 1 & -\varphi_{ji}^{-1}a \\ 0 & 1 \end{pmatrix}} \\
&= \begin{pmatrix} 1 & 0 \\ -b\varphi_{ji}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varphi_{ji} & 0 \\ b & -b\varphi_{ji}^{-1}a + c \end{pmatrix} \\
&= \begin{pmatrix} \varphi_{ji} & 0 \\ 0 & -b\varphi_{ji}^{-1}a + c \end{pmatrix} =: \begin{pmatrix} \varphi_{ji} & 0 \\ 0 & \tilde{c} \end{pmatrix}
\end{aligned}$$

A an isomorphism $\implies \phi_{ji}$ and $\tilde{c}$ isomorphisms.
$\implies \tilde{c} : \hat{M}_i \to \hat{N}_i$ isomorphism with $l(\hat{M}_i) < l(M)$. By induction the claim is true for M_i. Since $M_i \simeq N_j$, the claim follows

$\square$

9. Artin Algebras

Recall 9.1. (1) Λ finite dimensional k-algebra, k field, M finitely generated Λ-module $\implies \dim_k M = n < \infty$
$\implies \operatorname{End}_\Lambda(M) \subseteq \operatorname{End}_k(M) = M_n(k)$ finite dimensional k-algebra.
(2) Λ left artinian, but not right artinian, $M = {}_\Lambda \Lambda$. Then $\operatorname{End}_\Lambda(M) = \Lambda^{op}$ not left artinian.

R is a commutative ring.

Definition 9.2. (a) An <u>R-algebra</u> Λ is a ring Λ and a ring homomorphsim $\varphi : R \to \Lambda$ with $\operatorname{Im} \varphi \subseteq Z(\Lambda) = \{\lambda \in \Lambda \mid \lambda r = r\lambda, \forall r \in \Lambda\}$
(b) Λ_1, Λ_2 R-algebras given by $\varphi_1 : R \to \Lambda_1$ and $\varphi_2 : R \to \Lambda_2$. Then $\psi : \Lambda_1 \to \Lambda_2$ is a homomorphism of R-agebras, if ψ is a ring-homomorphism and
$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \Lambda_1 \arrow[d, "\psi"] \\ & \Lambda_2 \end{tikzcd}$$
commutes.
(c) Λ_1 is an R-subalgebra of Λ_2 if Λ_1 is a subring of Λ_2 and the inclusion $\Lambda_1 \hookrightarrow \Lambda_2$ is a homomorphism of R-algebras.

Note 9.3. Λ R-algebra $\implies \Lambda$ R-module.

Definition 9.4. Λ is an <u>artin R-algebra</u> if Λ is an R-algebra with R a commutative artinian ring and Λ is a finitelty generated R-module.

Example 9.5. (1) Any finitely generated algebra over a field k is an artin algebra.
(2) $k[x]$ is a k-algebra (k a field), but not an artin k-algebra ($\dim_k k[x] = \infty$).
(3) R a commutative artinian ring $\implies R$ an artin R-algebra ($\varphi = 1_R : R \to R$, $Z(R) = R$, R generated by 1 over R).

Note 9.6. (1) Λ R-algebra $\implies \Lambda^{op}$ R-algebra.
(2) A a Λ-module, Λ an R-algebra, $\varphi : R \to \Lambda \implies A$ is an R-algebra via
$$r \cdot a \overset{\text{def}}{=} \varphi(r)a, \quad \forall r \in R, \forall a \in A$$
(3) A, B Λ-modules, Λ an R-algebra ($\varphi : R \to \Lambda$)
$\implies \operatorname{Hom}_\Lambda(A, B)$ is an R-module via
$$f \in \operatorname{Hom}_\Lambda(A, B), r \in R \quad (r \cdot f)(a) \overset{\text{def}}{=} \varphi(r)f(a)$$
Check this as an exercise.

Proposition 38. *Λ artin R-algebra, finitely generated Λ-modules.*
(a) A, B Λ-modules $\implies \operatorname{Hom}_\Lambda(A, B)$ finitely generated R-module.
(b) A a Λ-module $\implies \operatorname{End}_\Lambda(A)$ is an artin R-algebra (which is an R-subalgebra of $\operatorname{End}_R(A)$).
(c) Λ is a left artinian ring.

Proof. (a) Have:
- $\cdot$ $\mathrm{Hom}_\Lambda(A,B) \subseteq \mathrm{Hom}_R(A,B)$
- $\cdot$ R artinian $\implies$ R noetherian

Enough to show that $\mathrm{Hom}_R(A,B)$ is a finitely generated R-module.
A finitely generated Λ-module $\implies$ $\exists \Lambda^n \to A$ onto Λ-homomorphism, $n \geq 1$.
Λ finitely generated R-module $\implies$ $\exists R^m \to \Lambda$ onto R-homomorphism, $m \geq 1$.
$\implies \exists\ R^{mn} \xrightarrow{\pi} A$ onto R-homomorphism.
$\implies$ $\mathrm{Hom}_R(A,B) \xrightarrow{\nu} \mathrm{Hom}_R(R^{mn},B) \simeq B^{mn}$

Exercise 9.7. (i) ν 1-1
(ii) $\mathrm{Hom}_R(R,B) \simeq B$ as R-modules ($f \mapsto f(1)$).
(iii) $\mathrm{Hom}_R(X \oplus Y, B) \simeq \mathrm{Hom}_R(X,B) \oplus \mathrm{Hom}_R(Y,B)$ as R-modules.

Induction: $\mathrm{Hom}_R(R^{mn},B) \simeq B^{mn}$ as R-modules
B^{mn} finitely generated R-module + R noetherian $\implies$ $\mathrm{Hom}_R(A,B)$ finitely generated R-module.

(b) $\mathrm{End}_R(A) = \mathrm{Hom}_R(A,A)$ is finitely genrated R-module by (a). R-algebra structure is given by $r \mapsto r \cdot 1_A$

(c) Λ finitely genrated R-module $\implies$ $l_R(\Lambda) < \infty$.
$\implies$ Λ artinian (and noetherian) R-module. Every left ideal in Λ is an R-submodule
$\implies$ Λ is left-artinian.

□

10. Categories and functors

Definition 10.1. A category $\mathcal{C}$ consist of a collection of objects, $\mathrm{Obj}\,\mathcal{C}$, and for each pair (A,B) in $\mathcal{C}$ a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$ (can be $\emptyset$), write $f : A \to B$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, and composition of morphisms

$$\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$

$$(g,f) \longmapsto gf$$

such that

(i) For each $A \in \mathcal{C}$, $\exists 1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ such that $f \cdot 1_A = f$, $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $1_A \cdot g = g$, $\forall g \in \mathrm{Hom}_{\mathcal{C}}(C,A)$.

(ii) Associative law is satisfied
$$h(gf) = (hg)f$$
when $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$.

Example 10.2.

(1) Γ a quiver, $J^t \subseteq \langle \rho \rangle \subseteq J^2$, $t \geq 2$.
$\mathrm{Rep}(\Gamma,\rho)$ = category of all representations of (Γ,ρ).
$\mathrm{Obj}(\mathrm{Rep}(\Gamma,\rho))$ = all representations of (Γ,ρ) over k.

morphisms = morphisms of representations.
composition = composition of morphisms of representations.

(2) Λ ring
$\operatorname{Mod}\Lambda$ = the category of all left Λ-modules.
$\operatorname{Obj}(\operatorname{Mod}(\Lambda))$ = all left Λ-modules.
morphisms = Λ-homomorphisms of left Λ-modules.
composition = usual composition of maps.
Special cases:
Ab = abelian groups = $\operatorname{Mod}(\mathbb{Z})$.
$\operatorname{Vec}(k)$ = vectorspaces over k = $\operatorname{Mod}(k)$.

Definition 10.3. $\mathcal{C}$ a category A, B objects in $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ is an isomorphism in $\mathcal{C}$ if $\exists$ a morphism $g : B \to A$ such that

$$gf = 1_A \text{ and } fg = 1_B.$$

Note 10.4. Λ a ring, $f : A \to B$ a Λ-homomorphism.

$$\begin{aligned} f \text{ isomorphism} &\overset{\text{def}}{\iff} f \text{ is bijective} \\ &\iff f \text{ is an isomorphism in } \operatorname{Mod}\Lambda. \end{aligned}$$

Definition 10.5. $\mathcal{C}$ a category. A category $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if $\operatorname{Obj}\mathcal{D} \subseteq \operatorname{Obj}\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{D}}(A, B) \subseteq \operatorname{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{D}$, and the composition in $\mathcal{D}$ is the restriction of the composition in $\mathcal{C}$.
$\mathcal{D}$ is a full subcategory of $\mathcal{C}$ if $\operatorname{Hom}_{\mathcal{D}}(A, B) = \operatorname{Hom}_{\mathcal{C}}(A, B)$ for $A, B \in \mathcal{D}$.

Note 10.6. Full subcategory - enough to describe the objects in the subcategory.

Example 10.7. (1) Λ not commutative: $\operatorname{Mod}\Lambda$ is a subcategory of $\operatorname{Mod}\mathbb{Z}$ which is not full:

$$\operatorname{Hom}_{\Lambda}(\Lambda, \Lambda) \subsetneq \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \Lambda)$$

Choose $z \notin Z(\Lambda)$, $f : \Lambda \to \Lambda$ by $f(\lambda) = z \cdot \lambda$. Then $f \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \Lambda)$, but $f \notin \operatorname{Hom}_{\Lambda}(\Lambda, \Lambda)$

(2) Λ a ring $I \subseteq \Lambda$ an ideal, $\pi : \Lambda \to \Lambda/I$ natural.
Any Λ/I-module M is also a Λ-module via $\lambda \cdot m \overset{\text{def}}{=} \pi(\lambda)m$

Exercise 10.8. $\operatorname{Mod}\Lambda/I \subseteq \operatorname{Mod}\Lambda$ is a full subcategory.
(Λ/I infinite representation type $\implies \Lambda$ infinite representation type.)

(3) Λ a ring, $M \in \operatorname{Mod}\Lambda$
$\operatorname{add} M =$ all direct summands in a finite number of coppies of M
($X \in \operatorname{add} M$; $M^n = X \oplus Y$).

Definition 10.9. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ associates to each object C in $\mathcal{C}$ an object $F(C)$ in $\mathcal{D}$, and to each morphsim $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$ such that

(i) $F(gf) = F(g)F(f)$ for all composable morphisms in $\mathcal{C}$
(ii) $F(1_A) = 1_{F(A)}$ $\forall A \in \operatorname{Obj}\mathcal{C}$

F contravariant functor:

$$f : A \to B \rightsquigarrow F(f) : F(B) \to F(A)$$

(i) $F(gf) = F(f)F(g)$ for all composable morphisms in $\mathcal{C}$
(ii) $F(1_A) = 1_{F(A)}$ $\forall A \in \operatorname{Obj}\mathcal{C}$

Example 10.10.

(1) $\Lambda = k\Gamma/\langle\rho\rangle$, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1, 2, \cdots, n\}$

$$\begin{aligned} F : \operatorname{mod}\Lambda &\longrightarrow \operatorname{Rep}(\Gamma, \rho) \\ M &\longmapsto F(M) = (V, f) \\ &\quad V(i) = \overline{e}_i M \\ &\quad \alpha : i \to j \in \Gamma_1 \\ &\quad f_\alpha : V(i) = \overline{e}_i M \xrightarrow{\overline{\alpha}\cdot -} \overline{e}_j M = V(j) \\ &\quad \overline{e}_i m \longmapsto \overline{\alpha}\cdot \overline{e}_i m \end{aligned}$$

$$\begin{tikzcd} M \arrow[d, "h"] & F(M) = (V, f) \arrow[d, "F(h)"] & V(i) = \overline{e}_i M \arrow[d, "F(h)(i) = h|_{\overline{e}_i M}"] \\ M' & F(M') = (V', f') & V'(i) = \overline{e}_i M' \end{tikzcd}$$

with $h \longmapsto F(h)$.

(2) $\mathcal{C}$ a category, $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$
$id_{\mathcal{C}}(C) = C$, $\forall C \in \operatorname{Obj}(\mathcal{C})$
$f : A \to B$ in $\mathcal{C}$
$id_{\mathcal{C}}(f) = f : id_{\mathcal{C}}(A) = A \to B = id_{\mathcal{C}}(B)$
$id_{\mathcal{C}}$ = identity functor.

(3) $A \in \operatorname{Mod}\Lambda$
$F = \operatorname{Hom}_\Lambda(A, -) : \operatorname{Mod}\Lambda \to Ab$
$F(B) = \operatorname{Hom}_\Lambda(A, B)$

$$\begin{array}{ccc} & F(B) & & F(C) \\ & \parallel & & \parallel \\ f : B \to C,\ F(f) : & \operatorname{Hom}_\Lambda(A, B) & \longrightarrow & \operatorname{Hom}_\Lambda(A, C) \\ & \cup & & \\ & g : A \to B & \longmapsto & F(g) = f \cdot g \end{array}$$

(4) Γ a quiver, $\operatorname{Obj}(\Gamma) = \Gamma_0$
morphsims $i \to j$ = all paths from i to j.
$F : \Gamma \to \operatorname{mod} k = \operatorname{vec} k \rightsquigarrow$ representations of Γ over k

Definition 10.11. $\mathcal{C}$ a category, R a commutative ring. $\mathcal{C}$ is <u>preadditive</u> (R-category) if $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group (R-module) for all objects A and B in $\mathcal{C}$, and the composition

$$\varphi : \operatorname{Hom}_{\mathcal{C}}(B, C) \times \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{C}}(A, C)$$

is bilinear (R-bilinear) for all A, B and C in $\mathcal{C}$, i.e

$$\varphi(g_1 + g_2, f) = \varphi(g_1, f) + \varphi(g_2, f)$$
$$\varphi(g, f_1 + f_2) = \varphi(g, f_1) + \varphi(g, f_2)$$
$$\Big(\varphi(g, rf) = \varphi(rg, f) = r\varphi(g, f)\Big)$$

Definition 10.12. $\mathcal{C}, \mathcal{D}$ preadditve ($R-$)categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an <u>additive</u> ($R-$)functor if the map

$$F : \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{D}}(F(A), F(B))$$

is a homomorphism of groups (R-modules) for all objects A and B in $\mathcal{C}$.

Example 10.13.

(1) $\Lambda = k\Gamma/\langle\rho\rangle$, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1, 2, \cdots, n\}$

Rep(Γ, ρ) - preadditive k-kategory

mod Λ - ————————"————————

$F : \operatorname{mod}\Lambda \to \operatorname{Rep}(\Gamma, \rho)$ additive k-functor

$H : \operatorname{Rep}(\Gamma, \rho) \to \operatorname{mod}\Lambda$ additive k-functor

$(V, f) \mapsto H(V, f) = V(1) \oplus V(2) \oplus \cdots \oplus V(n)$

$\overline{e}_i \cdot (v_1, v_2, \cdots, v_n) = (0, 0, \cdots, 0, v_i, 0, \cdots, 0)$

$\alpha : i \to j \in \Gamma$

j-th coord

$\overline{\alpha} \cdot (v_1, v_2, \cdots, v_n) = (0, 0, \cdots, 0, \ f_\alpha(v_i), \ 0, \cdots, 0)$

$$\begin{tikzcd} (V,f) \arrow[d, "h"] \\ (V',f') \end{tikzcd} \qquad H(h) = h_1 \oplus h_2 \oplus \cdots \oplus h_n : \qquad \begin{tikzcd} H(V,f) \arrow[d] & = & V(1) \oplus V(2) \oplus \cdots \oplus V(n) \\ H(V',f') & = & V'(1) \oplus V'(2) \oplus \cdots \oplus V'(n) \end{tikzcd}$$

(2) $\mathcal{C}$ preadditive, $\mathcal{D} \subseteq \mathcal{C}$ a full subcategory $\implies \mathcal{D}$ preadditive.
In particular, Λ artin R-algebra Mod Λ, mod Λ, add M are R-categories
$\operatorname{Hom}_\Lambda(A, -) : \operatorname{mod}\Lambda \to \operatorname{mod}R$ additve R-functor.

10.1. **Morphisms of Functors.**

Definition 10.14. $\mathcal{C}, \mathcal{D}$ categories, $F, G : \mathcal{C} \to \mathcal{D}$ functors. Then $\varphi : F \to G$ is a morphism of functors if for each object $C \in \mathcal{C}$ there is a morphism $\varphi_C : F(C) \to G(C)$ in $\mathcal{D}$ such that for each morphism $f : A \to B$ in $\mathcal{C}$ there is a commutative diagram.

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"] & G(B) \end{tikzcd}$$

$\varphi : F \to G$ is an isomorphism if $\varphi_C : F(C) \to G(C)$ is an isomorphism in $\mathcal{D}$ for all C in $\mathcal{C}$.

Example 10.15.

(1) k field.

$$(-)^* = \operatorname{Hom}_k(-, k) : \operatorname{Vec}(k) \to \operatorname{Vec}(k)$$

Define $\varphi : id_{\operatorname{Vec}(k)} \to (-)^* = \operatorname{Hom}_k(\operatorname{Hom}_k(-, k), k)$ by

$$\begin{array}{ccccc} \varphi_V : id_{\operatorname{Vec}(k)}(V) & = & V \longrightarrow V^{**} & = & \operatorname{Hom}_k(\operatorname{Hom}_k(V, k), k) \\ & & \Cup & & \\ & & v \longmapsto \varphi_V(v) & & \end{array}$$

where $\varphi_V(v)(f) = f(v)$, $f \in \operatorname{Hom}_k(V, k)$. φ is a morphism of functors.

Exercise 10.16.

(i) φ_V is 1-1

(ii) $\dim_k(V) < \infty \implies \varphi_V$ isomorphism

(2) Γ a quiver, $F, G : \Gamma \to \operatorname{vec}(k)$ functors. What is a morphsim $\varphi : F \to G$?

Definition 10.17. $\mathcal{C}, \mathcal{D}$ (R-)categories, $F : \mathcal{C} \to \mathcal{D}$ an (R-)functor. Then $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of (R-)categories if $\exists$ an (R-)functor $H : \mathcal{D} \to \mathcal{C}$ such that $HF \simeq id_{\mathcal{C}}$ and $FH \simeq id_{\mathcal{D}}$.

Definition 10.18. $F : \mathcal{C} \to \mathcal{D}$ (R-)functor

(a) F is full if $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is onto for all A, B in $\mathcal{C}$
(b) F is faithfull if $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is 1-1 for all A, B in $\mathcal{C}$
(c) F is dense if for each object D in $\mathcal{D}$, $\exists$ an object C in $\mathcal{C}$ such that $F(C) \simeq D$.

It can be shown:

Proposition 39. *$\mathcal{C}, \mathcal{D}$ (R-)categories, $F : \mathcal{C} \to \mathcal{D}$ (R-)functor.*
F is an equivalance $\iff$ F is full, faithfull and dense

Theorem 40. (Γ, ρ) *quiver with relations, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Lambda = k\Gamma/\langle\rho\rangle$. The functors $F : \mathrm{mod}\,\Lambda \to \mathrm{Rep}(\Gamma, \rho)$ and $H : \mathrm{Rep}(\Gamma, \rho) \to \mathrm{mod}\,\Lambda$ are inverse equivalenses of k-categories. $HF \simeq id_{\mathrm{mod}\,\Lambda}$ and $FH \simeq id_{\mathrm{Rep}(\Gamma,\rho)}$*

Proof.

(1) F equivalense:

(i) F faithfull: $h : A \to B \in \mathrm{mod}\,\Lambda$, $\Gamma_0 = \{1, 2, \cdots, n\}$
$F(h) = \{h_i\}_{i=1}^n$, where $h_i = h_{|\bar{e}_i A} : \bar{e}_i A \to \bar{e}_i B$

$$\begin{aligned} F(h) = 0 \implies \quad h_i &= 0, \forall i = 1, 2, \cdots, n \\ \| \quad & \\ h_{|\bar{e}_i A} &= 0, \forall i = 1, 2, \cdots, n \end{aligned}$$

$$\Rightarrow h = \underset{\substack{\| \\ 0}}{h_{|\bar{e}_1 A}} \oplus \underset{\substack{\| \\ 0}}{h_{|\bar{e}_2 A}} \oplus \cdots \oplus \underset{\substack{\| \\ 0}}{h_{|\bar{e}_n A}} : \bar{e}_1 A \oplus \underset{\substack{\downarrow \\ B}}{\cdots} \oplus \bar{e}_n A$$

$\implies h = 0 \implies F$ is faithfull.

(ii) F dense:
Check: Given $(V, f) \in \mathrm{Rep}(\Gamma, \rho)$ then

$$FH(V, f) \simeq (V, f), \quad (V, f) \in \mathrm{Rep}(\Gamma, \rho)$$

$\implies F$ dense.

(iii) F full: Given $\{h_i\} : F(A) \to F(B)$ i.e. $h_i : \bar{e}_i A \to \bar{e}_i B$ for $i = 1, 2, \cdots, n$.
Know: $\tilde{h} = h_1 \oplus \cdots \oplus h_n : A \longrightarrow B$

$$\begin{array}{c} \| \\ \bar{e}_1 A \oplus \cdots \oplus \bar{e}_n A \longrightarrow \bar{e}_1 B \oplus \cdots \oplus \bar{e}_n B \end{array}$$

Check: $F(\tilde{h}) = \{h_i\} \implies F$ full.
$\implies F$ is an equivalence.

(2) F and H ar inverse equivalences:
Exercise.

$\square$

Can be shown:
$F : \mathrm{Mod}\,R \to \mathrm{Mod}\,S$ equivalence

· M (semi)simple, finitely generated, artinian, noetherian, indecomposable $\iff F(M)$ is
· $Z(R) \simeq Z(S)$

11. Projectivization

Λ artin R-algebra, $A \in \operatorname{mod} \Lambda$
Have seen: $\Gamma = \operatorname{End}_\Lambda(A)^{op}$ is an artin R-algebra.
A is a left $\operatorname{End}_\Lambda(A)$-module:
$f \in \operatorname{End}_\Lambda(A), a \in A$, then $f \cdot a \overset{\text{def}}{=} f(a)$ (Exercise)
$\implies A$ is a right Γ-module.

Exercise 11.1.

· ${}_\Lambda A_\Gamma$ is a $\Lambda - \Gamma$-bimodule
  - left Λ-module
  - right Γ-module
  - $(\lambda \cdot a)\gamma = \lambda(a \cdot \gamma)$

· $\operatorname{Hom}_\Lambda({}_\Lambda A_\Gamma, X)$ is a left Γ module, ($X \in \operatorname{mod}\Lambda$) by
$f \in \operatorname{Hom}_\Lambda({}_\Lambda A_\Gamma, X)$, $g \in \Gamma$; $(g \cdot f)(a) \overset{\text{def}}{=} f(g(a))$, $\forall a \in A$

$\implies$ Have R-functor

$$F = \operatorname{Hom}_\Lambda(A, -) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$$

Lemma 41. Λ *ring*

(a)

$$\operatorname{Hom}_\Lambda(A, B_1 \oplus B_2) \overset{\alpha}{\longleftarrow} \operatorname{Hom}_\Lambda(A, B_1) \oplus \operatorname{Hom}_\Lambda(A, B_2)$$

given by $\alpha(f,g)(a) \overset{\text{def}}{=} (f(a), g(a))$ *for* $a \in A$, *is an isomorphism.*

(b) $\operatorname{Hom}_\Lambda(A_1 \oplus A_2) \simeq \operatorname{Hom}_\Lambda(A_1, B) \oplus \operatorname{Hom}_\Lambda(A_2, B)$
In particular, if $\Gamma = \operatorname{End}_\Lambda(A)^{op}$, *then the isomorphism* α *in (a) is an isomorphism of* Γ*-modules. Similarly in (b).*

Proof. Exercise. $\square$

Proposition 42. Λ *artin* R*-algebra,* $A \in \operatorname{mod}\Lambda$, $\Gamma = \operatorname{End}_\Lambda(A)^{op}$

$$e_A = \operatorname{Hom}_\Lambda(A, -) : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma \quad (R\text{-}functor)$$

has the following properties:

(a) $e_A : \operatorname{Hom}_\Lambda(Z, X) \to \operatorname{Hom}_\Gamma(e_A(Z), e_A(X))$ *is an* R*-isomorphism for all* Z *in* $\operatorname{add} A$ *(i.e.* $\exists Y$ *such that* $X \oplus Y \simeq A^t$ *for some* $t \geq 1$*)*
(b) $X \in \operatorname{add} A \implies e_A(X)$ *is a projective* Γ*-module*
(c) $e_{A|\operatorname{add} A} : \operatorname{add} A \to \mathcal{P}(\Gamma) = \{$*finitely generated projective* Γ*-modules*$\}$ *is an equivalence of* R*-categories.*

Proof.

(a) (i) $e_A : \operatorname{Hom}_\Lambda(A, X) \xrightarrow{\alpha} \operatorname{Hom}_\Gamma(e_A(A), e_A(X))$

$$\begin{array}{ccccc} \cup & & & & \\ f & \longmapsto & \alpha(f) = f_* : & {}_\Lambda(A,A) \to {}_\Lambda(A,X) \\ & & & g \longmapsto f \cdot g \end{array}$$

is an isomorphism of R-modules.

- α is an R-homomorphism: Exercise.
- <u>α is 1-1:</u> Assume that $\alpha(f) = f_* = 0$
  $\implies f_*(g) = f \circ g = 0$, $\forall g \in \operatorname{Hom}_\Lambda(A, A)$.
  Choose $g = 1_A \implies f = 0 \implies \alpha$ is 1-1
- <u>α is onto:</u> Given $h : \operatorname{Hom}_\Lambda(A, A) \to \operatorname{Hom}_\Lambda(A, X)$.
  Let $f = h(1_A) : A \to X \in \operatorname{Hom}_\Lambda(A, X)$. Then for $g \in \operatorname{Hom}_\Lambda(A, A)$
  $\alpha(f)(g) = fg = h(1_A) \cdot g$ (action of Γ on $\operatorname{Hom}_\Lambda(A, X)$)
  h a Γ-homomorphism $\implies h(1_A) \cdot g = h(1_A \cdot g) = h(g)$
  $\implies \alpha(f) = h \implies \alpha$ is onto.

(ii) <u>e_A "additive":</u> The following diagram commutes, $M \xrightarrow{i_M} M \oplus N \xleftarrow{i_N} N$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[dddd, mapsto, bend right] & f_* \arrow[ddr, mapsto, bend left, "Lemma41"] & \\
e_A : \operatorname{Hom}_\Lambda(M \oplus N, X) \arrow[r] \arrow[dd, "\wr"] & \operatorname{Hom}_\Gamma(e_A(M \oplus N), e_A(X)) \arrow[d, "\wr"] & \\
 & \operatorname{Hom}_\Gamma(e_A(M) \oplus e_A(N), e_A(X)) \arrow[d, "\wr"] & (fi_M)_* \oplus (fi_N)_* \arrow[ddl, mapsto, bend left] \\
e_A \oplus e_A : \operatorname{Hom}_\Lambda(M, X) \oplus \operatorname{Hom}_\Lambda(N, X) \arrow[r] & \operatorname{Hom}_\Gamma(e_A(M), e_A(X)) \oplus \operatorname{Hom}_\Gamma(e_A(N), e_A(X)) & \\
(fi_M, fi_N) \arrow[r, mapsto] & ((fi_M)_*, (fi_N)_*) &
\end{tikzcd}$$

(iii) <u>$Z = A^n$:</u> Diagram commutes by (ii)

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(A^n, X) \arrow[r, "e_A"] \arrow[d, "\wr"] & \operatorname{Hom}_\Gamma(e_A(A^n), e_A(X)) \arrow[d, "\wr"] \\
\operatorname{Hom}_\Lambda(A, X)^n \arrow[r, "\overbrace{e_A \oplus \cdots \oplus e_A}^{n}"] & \operatorname{Hom}_\Gamma(e_A(A), X)^n
\end{tikzcd}$$

isomorphism by (i)

$\implies e_A$ is an isomorphism for $Z = A^n$

(iv) <u>$Z \in \operatorname{add} A$, $Z \oplus Y \simeq A^n$, $n \geq 1$:</u>

diagram comutes by (ii)

isomorphism by (iii)

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(Z \oplus Y, X) \arrow[r, "e_A \sim"] \arrow[d, "\wr"] & \operatorname{Hom}_\Gamma(e_A(Z \oplus Y), e_A(X)) \arrow[d, "\wr"] \\
\operatorname{Hom}_\Lambda(Z, X) \oplus \operatorname{Hom}_\Lambda(Y, X) \arrow[r, "e_A \oplus e_A"] & \operatorname{Hom}_\Gamma(e_A(Z), e_A(X)) \oplus \operatorname{Hom}_\Gamma(e_A(Y), e_A(X))
\end{tikzcd}$$

$\implies e_A \oplus e_A$ is an isomorphism
$\implies e_A : \operatorname{Hom}_\Lambda(Z, X) \to \operatorname{Hom}_\Gamma(e_A(Z), e_A(X))$ is an isomorphism for all $Z \in \operatorname{add} A$ and $X \in \operatorname{mod} \Lambda$.

(b) Let $X \in \operatorname{add} A$, i.e. $\exists n \geq 1$ such that $A^n \simeq X \oplus Y$ for some Y
$\implies$

$$\begin{array}{ccccccc} e_A(A^n) & = & \mathrm{Hom}_\Lambda(A, A^n) & \simeq & \mathrm{Hom}_\Lambda(A,A)^n & \simeq & \Gamma^n \\ \wr\| & & & & & & \\ e_A(X \oplus Y) & = & \mathrm{Hom}_\Lambda(A, X \oplus Y) & \simeq & \mathrm{Hom}_\Lambda(A,X) \;\oplus\; \mathrm{Hom}_\Lambda(A,Y) & & \\ & & & & \| & & \\ & & & & e_A(X) & & \end{array}$$

as Γ-modules

$\implies e_A(X)$ is a projective Γ-module.

(c) (b) $\implies e_A : \operatorname{add} A \to \mathcal{P}(\Gamma)$
(a) $\implies e_A$ is full and faithfull

<u>e_A dense:</u> Let $P \in \mathcal{P}(\Gamma)$. Then $\exists n \geq 1$ and a Q suc that $\Gamma^n \xrightarrow[\sim]{\varphi} P \oplus Q$
Let $f' : P \oplus Q \to P \oplus Q$ be given by $f'(p,q) = (0,q)$. Then $(f')^2 = f'$ and $\operatorname{Ker} f' \simeq P$, and

$$\begin{array}{llcc} & & \Gamma^n & \\ & & \wr\uparrow\sigma & \\ e_A : \mathrm{Hom}_\Lambda(A^n, A^n) \xrightarrow{\sim} & \mathrm{Hom}_\Gamma(e_A(A^n),\; e_A(A^n)) & & g \\ & & & \downarrow \\ & \xrightarrow[\sim]{\psi} \mathrm{Hom}_\Gamma(\Gamma^n, \Gamma^n) & & \sigma g \sigma^{-1} \\ & \quad\quad \cup & & \\ & \quad\;\; \varphi^{-1} f' \varphi & & \end{array}$$

Choose $u : A^n \to A^n$ such that
$$\psi e_A(u) = \varphi^{-1} f' \varphi : \Gamma^n \to \Gamma^n$$
Let $f = \varphi^{-1} f' \varphi$. Note that $f^2 = f$. We have an exact sequence
$$0 \longrightarrow \operatorname{Ker} u \longrightarrow A^n \xrightarrow{u} A^n$$
This induces the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & e_A(\operatorname{Ker} u) \arrow[r] & e_A(A^n) \arrow[r, "e_A(u)"] & e_A(A^n) \\ 0 \arrow[r] & \operatorname{Ker} f \arrow[r] \arrow[u, dotted, "\wr"', "\Leftarrow"] \arrow[d, dotted, "\wr"', "\Leftarrow"] & \Gamma^n \arrow[r, "\psi e_A(u) = f"] \arrow[u, "\wr\sigma"] \arrow[d, "\wr\varphi"] & \Gamma^n \arrow[u, "\wr\sigma"] \arrow[d, "\wr\varphi"] \\ 0 \arrow[r] & P \arrow[r] & P \oplus Q \arrow[r, "f'"] & P\oplus \end{tikzcd}$$

Problem sheets $\implies e_A(\operatorname{Ker} u) \simeq \operatorname{Ker} f \simeq P$
<u>$\operatorname{Ker} u \in \operatorname{add} A$:</u> $f = \psi e_A(u) = f^2 = (\psi e_A(u))^2 = \psi(e_A(u)^2) = \psi e_A(u^2)$

$\implies u = u^2$ since ψ is an isomorphism and e_A is full and faithfull.

$$(A^n \in \operatorname{add} A$$

$\implies A^n = \operatorname{Ker} u \oplus \operatorname{Im} u \implies \operatorname{Ker} u \in \operatorname{add} A$
$\implies P \in e_A(\operatorname{add} A) \implies e_{A|\operatorname{add} A}$ is dense.

$\square$

Lemma 43. Λ *artin R-algerba, $A \in \operatorname{mod} \Lambda$, $\Gamma = \operatorname{End}_\Lambda(A)^{op}$*

$$e_A : \operatorname{add} A \to \mathcal{P}(\Gamma)$$

(a) $X \neq (0)$ *in* $\operatorname{add} A \iff e_A(X) \neq (0)$ *in* $\mathcal{P}(\Gamma)$, *for* $X \in \operatorname{add} A$
(b) $X \in \operatorname{add} A$
X *is indecomposable* $\iff e_A(X)$ *is indecomposable*
(c) $X, Y \in \operatorname{add} A$
$e_A(X) \simeq e_A(Y) \iff X \simeq Y$

Proof.
(a) Proposition 42 $\implies \operatorname{Hom}_\Lambda(X, X) \simeq \operatorname{End}_\Gamma(e_A(X), e_A(X))$ for $X \in \operatorname{add} A$
$\Leftarrow$: $e_A(X) = (0) \implies \operatorname{Hom}_\Lambda(X, X) = 0 \implies 1_X = 0 \implies X = (0)$
$\Rightarrow$: $X = (0) \implies e_A(X) = \operatorname{Hom}_\Lambda(A, (0)) = (0)$
(b) Have: $\operatorname{End}_\Lambda(X) \simeq \operatorname{End}_\Gamma(e_A(X))$ as rings
Follows from this.
(c) Exercise: General property of and equivalence.

$\square$

12. Basic Artin Algebras

Definition 12.1. Λ artin R-algebra. Then Λ is basic if

$$\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$$

with P_i indecomposable, then $P_i \not\simeq P_j$ for $i \neq j$.

Recall 12.2. $\Lambda/\mathfrak{r} = S_1 \oplus \cdots S_n$ - semisimplie, S_i simple.
$P(S_i) \to S_i$ projective cover $\implies \Lambda \simeq P(S_1) \oplus \cdots \oplus P(S_n)$
We have: $P(S_i) \simeq P(S_j) \iff S_i \simeq S_j$

Λ basic $\iff S_i \not\simeq S_j$ for $i \neq j$

Example 12.3.
(1) $\Lambda = \left(\begin{smallmatrix} k & k \\ k & k \end{smallmatrix}\right)$, k a field

$$\Lambda = \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & k \\ 0 & k \end{smallmatrix}\right)$$

simple Λ-modules $\Longrightarrow$ indecomposable

$\left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right) \xrightarrow{-\cdot\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} \left(\begin{smallmatrix} 0 & k \\ 0 & k \end{smallmatrix}\right)$ is an isomorphism of left Λ-modules
$\implies \Lambda$ is not basic.
(2) (Γ, ρ) a quiver with relations k a field, $J^t \subseteq \langle \rho \rangle \subseteq J^2$, $\Gamma_0 = \{1, 2, \cdots, n\}$.
$\Lambda = k\Gamma/\langle \rho \rangle$ is basic:
$\Lambda/\mathfrak{r} \simeq k\bar{e}_1 \oplus k\bar{e}_2 \oplus \cdots \oplus k\bar{e}_n$ with $k\bar{e}_i \not\simeq k\bar{e}_j$ for $i \neq j$
$\implies \Lambda$ basic.

Note 12.4. Λ artin R-algebra
<u>We know:</u> $\Lambda \simeq P_1^{n_1} \oplus P_2^{n_2} \oplus \cdots \oplus P_t^{n_t}$, P_i indecomposable, $P_i \not\simeq P_j$ for $i \neq j$
Λ basic $\iff n_i = 1$ for all $i = 1, 2, \cdots, t$.
Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_t$, and let $\Sigma = \mathrm{End}_\Lambda(P)^{op}$. Have that

$$e_P = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Sigma$$

where

$$e_P(P) = \Sigma \simeq e_P(P_1) \oplus e_P(P_2) \oplus \cdots \oplus e_P(P_t)$$

indec since P_i is indec

$e_P(P_i) \not\simeq e_P(P_j)$ for $i \neq j$, since $P_i \not\simeq P_j$ for $i \neq j \implies \Sigma$ basic.

Note 12.5. Λ basic then $P = \Lambda$ and

$$\Sigma = \mathrm{End}_\Lambda(\Lambda)^{op} \simeq \Lambda$$

Proposition 44. *Λ and Σ as above. Then*

$$e_P = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Sigma$$

is an equivalence of R-categories.

Proof. <u>Know:</u> e_P is an R-functor.
Proposition 42 (c) $\implies e_{P|\mathrm{add}\,P} : \mathrm{add}\,P \to \mathcal{P}(\Sigma)$ is an equivalence.

<u>e_P dense:</u> Let $C \in \mathrm{mod}\,\Sigma$. Then $\exists$ exact sequence $Q_1 \xrightarrow{f} Q_0 \to C \to 0$ with $Q_1, Q_0 \in \mathcal{P}(\Sigma)$. Since $Q_1 \xrightarrow{f} Q_0 \in \mathcal{P}(\Sigma)$ $\exists Q_1', Q_0' \in \mathrm{add}\,P$ and $f' : Q_1' \to Q_0' \in \mathrm{add}\,P$, such that $e_P(Q_i') \simeq Q_i$ and the following diagram commutes

$$\begin{tikzcd}
e_P(Q_1') \arrow[r, "e_P(f')"] \arrow[d, phantom, "\wr"] & e_P(Q_0') \arrow[r] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\Rightarrow"] & e_P(\mathrm{Coker}\, f') \arrow[r] \arrow[d, dotted, "\wr"] & 0 \\
Q_1 \arrow[r, "f"] & Q_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$Q_1' \xrightarrow{f'} Q_0' \longrightarrow \mathrm{Coker}\, f' \longrightarrow 0$ exact.

Exercise 12.6. P projective $\implies$
$e_P(Q_1') \xrightarrow{e_P(f')} e_P(Q_0') \longrightarrow e_P(\mathrm{Coker}\, f') \longrightarrow 0$ exact.

Problem sheets $\implies e_P(\mathrm{Coker}\, f') \simeq C \implies e_P$ dense.

<u>e_P full and faithfull:</u> Let $X, Y \in \mathrm{mod}\,\Lambda$, $\exists$ $\eta' : Q_1' \longrightarrow Q_0' \longrightarrow X \longrightarrow 0$ exact in $\mathrm{mod}\,\Lambda$ with $Q_i' \in \mathrm{add}\,P = \mathcal{P}(\Lambda)$
$\implies \eta : e_P(Q_1') \longrightarrow e_P(Q_0') \longrightarrow e_P(X) \longrightarrow 0$ exact.

Apply $\mathrm{Hom}_\Lambda(-,Y)$ to η' :

Prop $42(c)$

Problem sheets $\}$ exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(X,Y) \arrow[r] \arrow[d, "e_P"', "\wr"] \arrow[dr, phantom, "\Leftarrow"] & \mathrm{Hom}_\Lambda(Q'_0,Y) \arrow[r] \arrow[d, "e_P"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_\Lambda(Q'_1,Y) \arrow[d, "e_P"', "\wr"] \\
0 \arrow[r] & \mathrm{Hom}_\Gamma(e_P(X),e_P(Y)) \arrow[r] & \mathrm{Hom}_\Gamma(e_P(Q'_0),e_P(Y)) \arrow[r] & \mathrm{Hom}_\Gamma(e_P(Q'_1),e_P(Y))
\end{tikzcd}$$

$\implies e_P$ full and faithfull $\implies e_P : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Sigma$ is an equivalence. $\square$

Definition 12.7. Λ, Σ two rings. Then Λ and Σ are <u>Morita equivalent</u> if $\mathrm{Mod}\,\Lambda$ and $\mathrm{Mod}\,\Sigma$ are equivalent categories.
$\left(\begin{smallmatrix}\text{Anderson \& Fuller}\\ \text{page 266 Exer. 4}\end{smallmatrix}\right)$: (or $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Sigma$ are equivalent categories)

Theorem 45. *Let Λ be a finite dimensional k-algebra over an algebraically closed field k.*

(a) $\exists$ a basic k-algebra Σ such that Λ and Σ are Morita equivalent.
(b) Suppose that Λ is basic. Then $\exists(\Gamma,\rho)$ a quiver with relations over k such that $\Lambda \simeq k\Gamma/\langle\rho\rangle$, where $J^t \subseteq \langle\rho\rangle \subseteq J^2$ for some $t \geq 2$.

Proof. (a) Proposition 44
(b) (1) <u>Claim:</u> $\Lambda/\mathfrak{r} \simeq k^s$ for some $s \geq 1$.
<u>Know:</u> $\Lambda/\mathfrak{r}$ is semisimple $\implies \Lambda/\mathfrak{r} \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, D_i division ring.

$$e_{1,1} = diag(1,0,\cdots,0), e_{2,2} = diag(0,1,\cdots,0), e_{n_1,n_1} = diag(0,\cdots,0,1) \in M_{n_1}(D_1)$$

-complete set of primitive orthogonal idempotents of $M_{n_1}(D_1)$. Similarly for the other $M_{n_i}(D_i)$. Let

$$f_{ij} = (0,\cdots,0,\ \underset{\uparrow\ i\text{th coordinate}}{e_{jj}},\ 0,\cdots,0) \qquad ,j = 1,2,\cdots,n_i$$

$\implies \bigcup\limits_{i=1}^{r}\{f_{ij}\}_{j=1}^{n_i}$ complete set of orthogonal idempotents for $\Lambda/\mathfrak{r}$
We have $\Lambda/\mathfrak{r}f_{ij} \simeq \Lambda/\mathfrak{r}f_{ij'}$ $\forall j,j' \in \{1,2,\cdots,n_i\}$.
Lift $\bigcup\limits_{i=1}^{r}\{f_{ij}\}_{j=1}^{n_i}$ to Λ; $\bigcup\limits_{i=1}^{r}\{\tilde{f}_{ij}\}_{j=1}^{n_i}$, complete set of primitive orthogonal idempotents in Λ.
<u>Know:</u> $\Lambda\tilde{f}_{ij} \to \Lambda\tilde{f}_{ij}/\mathfrak{r}\tilde{f}_{ij} \simeq \Lambda/\mathfrak{r}f_{ij}$ projective cover.
$\implies \Lambda\tilde{f}_{ij} \simeq \Lambda\tilde{f}_{ij'}$, $\forall j,j' \in \{1,2,\cdots,n_i\}$.
Λ basic $\implies n_i = 1$ for all $i = 1,2,\cdots,r$.

$$\implies \Lambda/\mathfrak{r} \simeq D_1 \times D_2 \times \cdots \times D_r$$

(2) <u>Claim:</u> $D_i \simeq k$ for all $i = 1,2,\cdots,r$.

$$\dim_k(\Lambda) < \infty \implies \dim_k(\Lambda/\mathfrak{r})z\infty \implies d_i = \dim_k(D) < \infty \quad \forall i = 1,2,\cdots,r$$

<u>We have:</u> $k \overset{\nu}{\hookrightarrow} D_i$, $k \simeq \nu(k) \overset{\nu}{\hookrightarrow} D_i$

Suppose $D_i \setminus \nu(k) \neq \emptyset$. Let $z \in D_i \setminus \nu(k)$. Then $\{1, z, z^2, \cdots, z^{d_i}\}$ is linearly dependent.

$\implies \exists a_i \in \nu(k), i = 0, 1, 2, \cdots, d_i$ such that

$$a_0 \cdot 1 + a_1 z + a_2 z^2 + \cdots + a_{d_i} z^{d_i} = 0$$

that is, z is a root in the polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{d_i} x^{d_i}$$

$k \simeq \nu(k)$ - algebraically closed $\implies z \in \nu(k)$ Contradiction!

$\implies D_i = \nu(k) \simeq k$ for all i.

$\implies \Lambda/\mathfrak{r} \simeq k^r \ni f_i = (0, \cdots, 0, \underset{i\text{th coordinate}}{1}, 0, \cdots, 0)$, complete set of primitive orthogonal idempotents

- Lift $\{f_i\}_{i=1}^r$ to a complete set of primitive orthogonal idempotents $\{v_i\}_{i=1}^r$ in Λ
- Choose a basis $B_{ji} = \{a_{ji}(l)\}_l$ of $v_j\mathfrak{r}/\mathfrak{r}^2 v_i$ for all $i, j \in \{1, 2, \cdots, r\}$.
- Lift the elements in each B_{ji} to elements in $v_j \mathfrak{r} v_i$, $\tilde{B_{ji}} = \{\tilde{a}_{ji}(l)\}_l$
- Define Γ, $\Gamma_0 = \{i\}_{i=1}^r$,

$$\Gamma_1 = \{\alpha_{ji}(l) : i \to j \mid l = 1, 2, \cdots, \dim_k(v_j\mathfrak{r}/\mathfrak{r}^2 v_i)\}$$

- Define $\varphi : k\Gamma \longrightarrow \Lambda$ by letting

$$\begin{aligned} \varphi(e_i) &= v_i \\ \varphi(\alpha_{ji}(l)) &= \tilde{a}_{ji}(l) \end{aligned}$$

Problem sheets $\implies \varphi : k\Gamma \to \Lambda$ is a k-algebra homomorphism.

(3) <u>$\operatorname{Ker}\varphi$ is admissable:</u> Suppose that

$$(*) \qquad \varphi\left(\underbrace{\sum_i \gamma_i(0)e_i + \sum_{r,s,l} \gamma_{r,s,l}(1)\alpha_{rs}(l) + \text{longer paths}}_{=x}\right) = 0$$

(i) $(*) \implies \varphi(x) + \mathfrak{r} = 0$

$$= \sum_i \gamma_i(0)v_i + \underbrace{\text{"}\langle \tilde{a}_{ji}(l) \mid j, i, l\rangle\text{"}}_{\subseteq \mathfrak{r}} + \mathfrak{r} = 0$$

$$\implies \sum_i \gamma_i(0)v_i + \mathfrak{r} = 0$$

$$\implies \sum_i \gamma_i(0)f_i = 0 \text{ in } \Lambda/\mathfrak{r} \simeq k^r$$

$\{f_i\}$ a basis for $\Lambda/\mathfrak{r} \simeq k^r \implies \gamma_i(0) = 0, \forall i = 1, 2, \cdots, r$

(ii) $(*) \implies \varphi(x) + \mathfrak{r}^2 = 0$

$$= \sum_{r,s,l} \gamma_{r,s,l}(1)\tilde{a}_{rs}(l) + \underbrace{\text{"}\langle \tilde{a}_{rs}(l)\tilde{a}_{r's'}(l')\rangle\text{"}}_{\subseteq \mathfrak{r}^2} + \mathfrak{r}^2 = 0$$

$$\implies \sum_{r,s,l} \gamma_{r,s,l}(1)\tilde{a}_{rs}(l) + \mathfrak{r}^2 = 0$$

$$\implies \quad \textstyle\sum_{r,s,l} \gamma_{r,s,l}(1) \quad a_{rs}(l) \quad + \quad \mathfrak{r}^2 = 0 \text{ in } \quad \mathfrak{r}/\mathfrak{r}^2$$

basis for

$\implies \gamma_{rsl}(1) = 0,\ \forall r, s, l \implies x \in J^2$
$\implies \operatorname{Ker}\varphi \subseteq J^2$
We have: $\alpha_{rs}(l) \mapsto \tilde{a}_{rs}(l) \in \mathfrak{r}$, $\mathfrak{r}^m = (0)$ for some $m \geq 1$
$\implies$ {all paths of length $\geq m$} $\subseteq \operatorname{Ker}\varphi$
$\implies J^m \subseteq \operatorname{Ker}\varphi$
$\operatorname{Ker}\varphi$ is admissable.

(4) φ onto: Let $\lambda \in \Lambda$. Then $\exists \gamma_i(0) \in k$ for $i = 1, 2, \cdots, r$ such that

$$x_1 = \lambda - \sum \gamma_i(0) v_i \in \mathfrak{r}$$

since $\{\overline{v}_i = v_i + \mathfrak{r}\}$ is a basis for $\Lambda/\mathfrak{r}$.
Can show: $\mathfrak{r}$ is generated by $\{\tilde{a}_{ji}(l)\}_{j,i,l}$ and $\tilde{a}_{ji}(l)$ is in the image of φ
$\implies \varphi$ is onto.
$\implies k\Gamma/\operatorname{Ker}\varphi \simeq \operatorname{Im}\varphi = \Lambda$

□

13. Duality

Definition 13.1. $\mathcal{C}$, $\mathcal{D}$ (R-)categories, $F : \mathcal{C} \to \mathcal{D}$ a contravariant (R-)functor. Then F is a duality if there exists a contravariant (R-)functor $H : \mathcal{D} \to \mathcal{C}$ such that

$$HF \simeq id_{\mathcal{C}} \text{ and } FH \simeq id_{\mathcal{D}}$$

as (R-)functors.

k a field, V a vector space

$$D := \operatorname{Hom}_k(-, k) : \operatorname{Vec}(k) \to \operatorname{Vec}(k)$$

is a contravariant functor. Assume that $\dim_k V = t < \infty$. If $\{v_i\}_{i=1}^t$ is a k-basis for V, then $\{v_i^*\}_{i=1}^t$ where $v_i^* \in D(V) = \operatorname{Hom}_k(V, k)$ is given by

$$v_i^*\left(\sum_{j=1}^t a_j v_j\right) = a_i, \quad a_j \in k$$

is a basis for $D(V)$ (the dual basis).
$\implies \dim_k V = \dim_k D(V) = \dim_k DD(V)$
Define φ_V by

$$\varphi_V : V \longrightarrow DD(V) = \operatorname{Hom}_k(\operatorname{Hom}_k(V, k), k)$$

$$v \longmapsto \varphi_V(v) : \operatorname{Hom}_k(V, k) \to k$$

where $\varphi_V(v)(f) = f(v)$ for $f \in D(V)$

Exercise 13.2.

(a) φ_V is 1-1 ($\Rightarrow \varphi_V$ is an isomorphism for $\dim_k V < \infty$)
(b) $\varphi = \{\varphi_V\}_{V \in \mathrm{vec}(k)} : id_{\mathrm{vec}(k)} \to DD$ is an isomorphism of functors.
Check:
(a) Λ finite dimensional k-algebra.
For $X \in \operatorname{mod}\Lambda$, then $D(X) = \operatorname{Hom}_k(X, k)$ is a left Λ^{op}-module via

$$(\lambda \cdot f)(x) = f(\lambda x)$$

for $f \in D(X)$, $\lambda \in \Lambda^{op}$ and $x \in X$.
(b) $f : X \to Y \in \operatorname{mod}\Lambda$. Then

$$\begin{tikzcd} D(f):\ D(Y) = \operatorname{Hom}_k(Y,k) \arrow[r] & \operatorname{Hom}_k(X,k) = D(X) \\ g \arrow[r, mapsto] & g \cdot f \end{tikzcd}$$

is a Λ^{op}-homomorphism.
$\Longrightarrow D = \operatorname{Hom}_k(-, k) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$
and $D = \operatorname{Hom}_k(-, k) : \operatorname{mod}\Lambda^{op} \to \operatorname{mod}\Lambda$

[Note: X finitely generated Λ-module
$\Longrightarrow \dim_k X < \infty$
$\Longrightarrow \dim_k D(X) < \infty$
$\Longrightarrow D(X)$ finitely generated Λ^{op}-module]

Proposition 46. *Λ finite dimensional k-algebra, k a field. Then $D = \operatorname{Hom}_k(-, k) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ is a duality.*

$\Lambda = k\Gamma/\langle\rho\rangle$ - a finite dimensional k-algebra, k a field. D induces a duality on representations.

$$\begin{tikzcd} \operatorname{Rep}(\Gamma,\rho) \ni & (V,f) \arrow[r, mapsto] \arrow[d, dotted] & F(V,f) \arrow[d, mapsto] & \in \operatorname{mod}\Lambda \\ \operatorname{Rep}(\Gamma^{op},\rho^{op}) \ni & H(D(F(V,f))) & DF(V,f) \arrow[l, mapsto] & \in \operatorname{mod}\Lambda^{op} \end{tikzcd}$$

(Γ, ρ), $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1, 2, \cdots, n\}$
Define (Γ^{op}, ρ^{op}) by

$\Gamma_0^{op} = \Gamma_0$
Γ_1^{op}: for each arrow $\alpha : i \to j$ in Γ_1, there is an arrow $\alpha^{op} : j \to i$ in Γ_1^{op}
If $p = \alpha_1\alpha_2 \cdots \alpha_{r-1}\alpha_r$ is a path in Γ, let
$p^{op} = \alpha_r^{op}\alpha_{r-1}^{op} \cdots \alpha_2^{op}\alpha_1^{op}$ be a path in Γ^{op}.

Then $\Lambda^{op} \simeq k\Gamma^{op}/\langle\rho^{op}\rangle$ and (Γ^{op}, ρ^{op}) is equivalent to $\operatorname{mod}\Lambda^{op}$.
Let $(V, f) \in \operatorname{Rep}(\Gamma, \rho)$. Then $F(V, f) \in \operatorname{mod}\Lambda$ and $DF(V, f) \in \operatorname{mod}\Lambda^{op}$. The underlying vector space of $DF(V, f)$ is $D(\bigoplus_{i=1}^{n} V(i))$
$HDF(V, F) = (V', f')$:

$$V'(i) = e_i^{op} DF(V, f) = e_i^{op} \operatorname{Hom}_k\left(\bigoplus_{j=1}^{n} V(j), k\right)$$

$$\nu_j : V_j \hookrightarrow \bigoplus_{l=1}^{n} V(l)\ , \qquad g \in \mathrm{Hom}_k\left(\bigoplus_{j=1}^{n} V(j), k\right)$$

Then

$$\begin{aligned} e_i^{op} g(v_1, v_2, \cdots, v_n) &= g(e_i(v_1, v_2, \cdots, v_n)) \\ &= g((0, 0, \cdots, 0, v_i, 0, \cdots, 0)) \\ &= g\nu_i(v_i) \end{aligned}$$

$$e_i^{op} g \rightsquigarrow g\nu_i \in D(V(i)) = \mathrm{Hom}_k(V(i), k)$$

$$V'(i) = D(V(i)) = \mathrm{Hom}_k(V(i), k)$$

$$DF(V, f) \ni g \mapsto g\nu_i$$

$$\alpha^{op} : j \to i$$

$$\begin{tikzcd}
h \arrow[d, phantom, "\parallel" description] & f \cdot f_\alpha \arrow[d, phantom, "\parallel" description] \\
g\nu_j \arrow[r, mapsto] & g\nu_j f_\alpha \\
V'(j) = D(V(j)) \arrow[r] \arrow[d, phantom, "\wr|" description] & V'(i) = D(V(i)) \arrow[d, phantom, "\wr|" description] \\
e_j^{op} DF(V, f) \arrow[r, "\alpha^{op} \cdot -"] & e_i^{op} DF(V, f) \\
g \arrow[r, mapsto] \arrow[uuuu, mapsto, bend left=60] & g\nu_j(f_\alpha(v_i))
\end{tikzcd}$$

$$\begin{aligned} (\alpha^{op} \cdot g)(v) &= g(\alpha \cdot v) = g(\alpha(v_1, v_2, \cdots, v_n)) \\ &= g((0, \cdots, 0, f_\alpha(v_i), 0, \cdots, 0)) \end{aligned}$$

j-th coordinate

$$= g\nu_j(f_\alpha(v_i))$$

$$\begin{aligned} &f'_{\alpha^{op}} \;:\; V'(j) = D(V(j)) \to D(V(i)) = V'(i) \\ &\quad\parallel \\ &D(f_\alpha) \end{aligned}$$

$$\Longrightarrow (V', f') = \Big(\{D(V(i))\}_{i=1}^{n}, \{D(f_\alpha)\}_{\alpha \in \Gamma_1}\Big).$$

Exercise 13.3. V, W finite dimensional vector spaces B, B' basis for V and W respectively. B^*, B'^* dual basis for $D(V)$ and $D(W)$.
Let $f : V \to W$. Suppose that

$$m_B^{B'}(f) = A$$

Then for $D(f) : D(W) \to D(V)$ we have the matrix representation

$$m_{B'^*}^{B^*}(D(f)) = A^T$$

for the dual map $D(f)$.

Example 13.4.

$$\Gamma: \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \end{tikzcd},\ \rho = \{\gamma\beta\},\ \Lambda = k\Gamma/\langle\rho\rangle,\ k \text{ a field.}$$

$$\Gamma^{op}: \begin{tikzcd} 1 & 2 \arrow[l, bend right, "\alpha^{op}"'] \arrow[l, bend left, "\beta^{op}"] & 3 \arrow[l, "\gamma^{op}"'] \end{tikzcd},\ \rho^{op} = \{\beta^{op}\gamma^{op}\}$$

$$\Lambda\bar{e}_1: \begin{tikzcd} k \arrow[r, bend left, "\binom{1}{0}"] \arrow[r, bend right, "\binom{0}{1}"'] & k^2 \arrow[r, "(1\ 0)"] & k \end{tikzcd}$$

$$\Lambda\bar{e}_2: \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & k \arrow[r, "1"] & k \end{tikzcd}$$

$$\Lambda\bar{e}_3: \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & 0 \arrow[r] & k \end{tikzcd}$$

$$D(\Lambda\bar{e}_1): \begin{tikzcd} k & k^2 \arrow[l, bend right, "(1\ 0)"'] \arrow[l, bend left, "(0\ 1)"] & k \arrow[l, "\binom{1}{0}"'] \end{tikzcd}$$

$$D(\Lambda\bar{e}_2): \begin{tikzcd} 0 & k \arrow[l, bend right] \arrow[l, bend left] & k \arrow[l, "1"'] \end{tikzcd}$$

$$D(\Lambda\bar{e}_3): \begin{tikzcd} 0 & 0 \arrow[l, bend right] \arrow[l, bend left] & k \arrow[l] \end{tikzcd}$$

Lemma 47. *Λ finite dimensional k-algebra, k a field.*

- *(a)* $\eta: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *exact in* $\operatorname{mod}\Lambda$
  $\iff 0 \to D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \to 0$ *exact in* $\operatorname{mod}\Lambda^{op}$.
- *(b)* *S simple Λ-module $\iff$ $D(S)$ simple Λ^{op}-module.*
- *(c)* $l(A) = l(D(A))$ *for* $A \in \operatorname{mod}\Lambda$

Proof. (Sketch of proof)

- (a) Use that η splits as a sequence of k-modules, that D preserves k-dimesnion and that $D^2 \simeq id_{\operatorname{mod}\Lambda}$.
- (b) Use (a).
- (c) Induction on length.

□

Given a statement S in a category $\mathcal{C}$ then <u>the dual statement S^*</u> is the statement about $\mathcal{C}$ reversing the direction of all morphisms and replacing all compositions $\alpha\beta$ of morphisms with $\beta\alpha$.

Example 13.5.

P projective:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists h"'] \arrow[d, "f"] & \\ \text{exact } B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd} \qquad \forall g, \forall f \implies \exists h: P \to B \text{ such that } gh = f$$

I injective:

$$\begin{tikzcd} & I & \\ \text{exact } B \arrow[ur, dotted, "\exists h"] & C \arrow[l, "g"'] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd} \qquad \forall g, \forall f \implies \exists h: B \to I \text{ such that } hg = f$$

14. Injective modules

Definition 14.1. Λ a ring, $I \in \operatorname{Mod}\Lambda$. Then I is <u>injective</u> if for every monomorphism $i : A \to B$ in $\operatorname{Mod}\Lambda$ and every homomorphism $f : A \to I$ there exists an $h : B \to I$ such that $hi = f$ i.e. the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists h"] \\
 & I &
\end{tikzcd}$$

Proposition 48. *Λ finite dimensional k-algebra, k a field, $P \in \operatorname{mod}\Lambda$.*

(a) P projective Λ-module $\iff$ $D(P)$ is injective Λ^{op}-module.
(b) Any Λ-module $M \in \operatorname{mod}\Lambda$ is a submodule of an injective Λ-module in $\operatorname{mod}\Lambda$.

Proof.

(a) <u>$\Rightarrow$:</u> P projective. Consider
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \\
 & D(P) &
\end{tikzcd}$$
in $\operatorname{Mod}\Lambda^{op}$.

Apply D:

$$\begin{tikzcd}
 & D^2(P) \arrow[dl, dashed, "\exists h"'] \arrow[d, "D(f)"] & \simeq & P & \arrow[l] \text{projective} \\
D(B) \arrow[r, "D(i)"] & D(A) \arrow[r] & 0 & &
\end{tikzcd}$$

$\Longrightarrow$

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "\alpha_A"'] \arrow[dd, "f"'] & & B \arrow[dr, "\alpha_B"] & \\
 & D^2(A) \arrow[rr, "D^2(i)"] \arrow[dd, "D^2(f)"] & & D^2(B) \arrow[ddll, "D(h)"] \\
D(P) \arrow[dr, "\alpha_{D(P)}"', "\sim"] & & & \\
 & D^3(P) & &
\end{tikzcd}$$

$\Longrightarrow \alpha_{D(P)} f = D(h)\alpha_B i$
$\alpha^{-1}_{D(P)} \cdot - \Big| \quad f = \left(\alpha^{-1}_{D(P)} D(h) \alpha_B\right) i$
$\Longrightarrow$ $D(P) \in \operatorname{mod}\Lambda^{op}$ is injective.

<u>$\Leftarrow$:</u> Baer criterion & Λ noetherian
$\Longrightarrow$ we can restrict ourselves to finitely generated modules. Use dual arguments.

(b) Let $M \in \operatorname{mod}\Lambda$. Then $D(M) \in \operatorname{mod}\Lambda^{op}$. Let $P \twoheadrightarrow D(M)$ be the projective cover of $D(M)$. Then

$$M \xrightarrow[\sim]{\alpha_M} D^2(M) \hookrightarrow D(P) \quad \in \quad \operatorname{mod}\Lambda$$

$D(P)$: injective Λ-module

□

Remark 14.2. Λ a ring, $M \in \operatorname{Mod}\Lambda$.
Can be shown: $M \hookrightarrow I$, I injective Λ-module.
However: Even if M is a finitely generated Λ-module I need not be a finitely generated Λ-module.

$$\Big(\mathbb{Z}\text{-modules} = Ab : \ \mathbb{Z} \hookrightarrow \mathbb{Q}\ \Big)$$

Definition 14.3. Λ a ring

(a) $A \subseteq X$ Λ-modules. Then A is an essential submodule of X, if for each non-zero submodule B of X, then $A \cap B \neq (0)$.
(b) A monmorphism $i : A \to X$ is essential if $i(A)$ is an essential submodule of X.
(c) A monomorphism $i : A \to I$ is an injective envelope if
  (i) I is injective
  (ii) i is an essential monomorphism.

Can be shown: (MA3204) Λ a ring. Every Λ-module has an injective envelope.

Definition 14.4. Λ artin R-algebra, $A \in \operatorname{mod}\Lambda$. The socle of A, $\operatorname{soc} A$ is the sum of all simple submodules of A.

Example 14.5. $\Gamma :$ $\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 & & 3 \end{tikzcd}$, k a field, $\Lambda = k\Gamma$

$$\Lambda e_1 \rightsquigarrow \begin{tikzcd}[column sep=small] & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\ k & & k \end{tikzcd} \hookleftarrow \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \arrow[dr] & \\ k & & 0 \end{tikzcd} + \begin{tikzcd}[column sep=small] & 0 \arrow[dl] \arrow[dr] & \\ 0 & & k \end{tikzcd} = \underbrace{\Lambda e_2 \oplus \Lambda e_3}_{\operatorname{soc}\Lambda e_1}$$

Note 14.6.

(1) In general, $\operatorname{soc} S = S$, when S is a (semi-)simple Λ-module.
(2) $\operatorname{soc} A \subseteq A$ is a semisimple submodule of A.
(3) Λ artin R-algebra, $A \in \operatorname{mod}\Lambda$, $A \neq (0)$.
Since $l(A) < \infty$, then $\exists$ a simple Λ-submodule $S \subseteq A$, that is, $\operatorname{soc} A \neq (0)$.
Furthermore $A \neq (0) \iff \operatorname{soc} A \neq (0)$, and $D(A/\mathfrak{r}A) \simeq \operatorname{soc} D(A)$.

Lemma 49. *Λ artin R-algebra, $A \subseteq X \in \operatorname{mod}\Lambda$*
TFAE

(a) *A essential submodule of X*
(b) $\operatorname{soc} X \subseteq A$
(c) $\operatorname{soc} A \subseteq \operatorname{soc} X$.

Proof. (a) ⇒ (c): $A \subseteq X$ essential submodule

$$
\begin{aligned}
S \subseteq X \text{ simple} &\implies S \neq (0) \\
&\implies (0) \neq S \cap A \subseteq S \\
\text{Simple} &\implies S \cap A = S \subseteq A \\
\implies \sum_{\substack{S \subseteq X \\ S \text{ simple}}} S &= \operatorname{soc} X \subseteq A \\
A \subseteq X &\implies \operatorname{soc} A \subseteq \operatorname{soc} X \\
\text{Hence, } \operatorname{soc} A &= \operatorname{soc} X
\end{aligned}
$$

(c) ⇒ (b): Obviously, $\operatorname{soc} A = \operatorname{soc} X$
$\implies \operatorname{soc} X \subseteq \operatorname{soc} A \subseteq A$.

(b) ⇒ (a): Assume that $\operatorname{soc} X \subseteq A$.
Let $(0) \neq B \subseteq X$. Then $(0) \neq \operatorname{soc} B \subseteq \operatorname{soc} X$
Hence, $(0) \neq \operatorname{soc} B = \operatorname{soc} B \cap \operatorname{soc} X \subseteq \operatorname{soc} B \cap A \subseteq B \cap A \implies B \cap A \neq (0)$. □

Proposition 50. *Λ artin R-algebra, $(0) \neq A \in \operatorname{mod} \Lambda$*

(a) $A \hookrightarrow I$ injective envelope $\iff$ I injective and $\operatorname{soc} A = \operatorname{soc} I$.
(b) Injective envelopes are unique up to isomorphism.
(c) Injective envelope of $A \simeq$ injective envelope of $\operatorname{soc} A$

Proof. (a) Use the definition of injective envelope and Lemma 49

(b) Let $A \overset{\nu_i}{\hookrightarrow} I_i$ be two injective envelopes

$$
\begin{tikzcd}
& 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\nu_1"] \arrow[dr, "\nu_2"'] & I_1 \arrow[d, dashed, "\exists\varphi"] \\
& & I_2
\end{tikzcd}
$$

Assume that $\operatorname{Ker} \varphi \neq (0)$
$\implies A \cap \operatorname{Ker} \varphi \neq (0)$
$\implies \exists a \in A \backslash \{0\}$ s.t. $0 = \varphi(a) = \varphi\nu_1(a) = \nu_2(a) \implies a = 0$ Contradiction!
$\implies \operatorname{Ker} \varphi = (0) \implies I_1 \simeq \underset{\text{injective}}{\operatorname{Im} \varphi} \subseteq I_2$

Recall 14.7. $0 \longrightarrow I \hookrightarrow B \longrightarrow C \longrightarrow 0$ exact, and I injective
$\implies B = I \oplus B'$

$\implies I_2 = \operatorname{Im} \varphi \oplus I_2'$
$\implies A \xhookrightarrow{\nu_2 = \varphi\nu_1} I_2 = \operatorname{Im} \varphi \oplus I_2'$ and $\operatorname{Im} \nu_2 \subseteq \operatorname{Im} \varphi\nu_1$
$\implies A \cap I_2' = (0)$
A essential submodule of I_2
$\implies I_2' = (0)$ and $\operatorname{Im} \varphi = I_2$
$\implies \varphi$ is an isomorphism and therefor injective envelopes are unique up to isomorphism.

(c) Consider

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{soc} A \arrow[r, hook, "\nu"] \arrow[d, hook, "\text{inj envelope } i"'] & A \arrow[dl, dashed, "{\exists f, \text{ since } I(\operatorname{soc} A) \text{ is inj.}}"] \\ & I(\operatorname{soc} A) & \end{tikzcd}$$

- $f\nu = i$ 1-1 $\implies \operatorname{Ker} f \cap \operatorname{soc} A = (0)$
  $\operatorname{soc} A \hookrightarrow A$ essential $\implies \operatorname{Ker} f = (0) \implies f$ 1-1.
- Let $(0) neq A' \subseteq I(\operatorname{soc} A)$. WTS: $f(A) \cap A' \neq (0)$
  We have: $f(A) \cap A' \supseteq \operatorname{soc} A \cap A' \neq (0)$, since $\operatorname{soc} A$ is an essential submodule of $I(\operatorname{soc} A)$
  $\implies f(A)$ is an essential submodule of $I(\operatorname{soc} A)$
  $\implies A \xrightarrow{f} I(\operatorname{soc} A)$ is an injective envelope

□

Lemma 51. Λ *artin R-algebra,* $A \in \operatorname{mod} \Lambda$*,* $\operatorname{rad} \Lambda = \mathfrak{r}$

$$\operatorname{soc} A = \{a \in A \mid \mathfrak{r} \cdot a = (0)\} = S_A$$

Proof. $\operatorname{soc} A$ semisimple $\implies \mathfrak{r} \operatorname{soc} A = (0) \implies \operatorname{soc} A \subseteq S_A$
S_A is a submodule of A.
$\mathfrak{r} S_A = (0) \implies S_A$ is a semisimple submodule
$\implies S_A \subseteq \operatorname{soc} A \implies \operatorname{soc} A = S_A$ □

Exercise 14.8. Λ artin R-algebra, $A, A_1, A_2 \in \operatorname{mod} \Lambda$

(a) $\operatorname{soc} A \simeq \operatorname{Hom}_\Lambda(\Lambda/\mathfrak{r}, A)$
(b) $\operatorname{soc}(A_1 \oplus A_2) = \operatorname{soc} A_1 \oplus \operatorname{soc} A_2$

Proposition 52. Λ *finite dimensional k-algebra, k a field.*

$P \xrightarrow{f} A$ *is a projective cover in* $\operatorname{mod} \Lambda$

$$\Updownarrow$$

$D(A) \xrightarrow{D(f)} D(P)$ *is a injective envelope in* $\operatorname{mod} \Lambda^{op}$

Proof. Use $X \in \operatorname{mod} \Lambda \implies \operatorname{soc} D(X) \simeq D(X/\mathfrak{r}X)$
Using duality one can show for Λ a finite dimensional k-algebra, k a field.

(a) $A, B \in \operatorname{mod} \Lambda \implies I(A \oplus B) = I(A) \oplus I(B)$
(b) I injective in $\operatorname{mod} \Lambda$
$$I \text{ indecomposable} \iff \operatorname{soc} I \text{ simple}$$
(c) There is a 1-1 correspondance between isomorphism classes of simple Λ-modules and isomorphism classes of indecomposable injective Λ-modules:

$$\begin{tikzcd}[column sep=small] & & \{\text{isomorphism classes of simple modules}\} \arrow[dd, leftrightarrow, "1-1"] \\ S \arrow[d, mapsto] & \operatorname{soc} I & \\ I(S) & I \arrow[u, mapsto] & \{\text{isomorphism classes of indecomposable injective modules}\} \end{tikzcd}$$

□

14.1. **The socle of a representation.** (Γ,ρ) quiver with relations ρ, k a field, $J^t \subseteq \langle\rho\rangle \subseteq J^2$, $\Gamma_0 = \{1,2,\cdots,n\}$. $\Lambda = k\Gamma/\langle\rho\rangle$, $\mathfrak{r} = \operatorname{rad}\Lambda = J/\langle\rho\rangle \subseteq \Lambda$

$$(V,f) \longmapsto F(V,f) = \bigoplus_{i=1}^{n} V(i)$$

$$\operatorname{Rep}(\Gamma,\rho) \underset{H}{\overset{F}{\rightleftarrows}} \operatorname{mod}\Lambda \qquad F(V,f) \supseteq \operatorname{soc} F(V,f) = \{m \in F(V,f) \mid \mathfrak{r}\cdot m = (0)\}$$

$$m = (v_1, v_2, \cdots, v_n) \in F(V,f)$$

$m \in \operatorname{soc} F(V,f) \iff \overline{\alpha}\cdot m = 0, \quad \forall\alpha : i \to j \in \Gamma$ since $\mathfrak{r} = J/\langle\rho\rangle$, $(J = \langle\text{arrows}\rangle)$

$\iff (0,\cdots,0,\ \underbrace{f_\alpha(v_i)}_{j\text{-th coord}},\ 0,\cdots,0) = 0, \forall\alpha : i \to j \in \Gamma$

$\iff f_\alpha(v_i) = 0, \forall\alpha : i \to j \in \Gamma$

$\iff v_i \in \bigcap_{\alpha: i\to j\in\Gamma} \operatorname{Ker} f_\alpha, \forall i = 1,2,\cdots,n$

$$H(\operatorname{soc} F(V,f)) = ?$$

Let $V'(i) = \bigcap_{\alpha:i\to j} \operatorname{Ker} f_\alpha \subseteq V(i)$ and let $f'_\alpha = f_\alpha |_{V'(i)} = 0$

Example 14.9.

$\Gamma :\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$, $\rho = \{\gamma\beta\}$, k a field, $\Lambda = k\Gamma/\langle\rho\rangle$

$$\Lambda\overline{e}_1 : \begin{tikzcd} k \arrow[d, bend right, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] \arrow[d, bend left, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \\ k^2 \arrow[d, "(0\ 1)"] \\ k \end{tikzcd} \qquad \Lambda\operatorname{soc}\overline{e}_1 : \begin{tikzcd} 0 \arrow[d, bend right, "0"'] \arrow[d, bend left, "0"] \\ k \oplus (0) \arrow[d, "0"] \\ k \end{tikzcd}$$

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References

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