Exercise 5. Consider the poset

$$(X, \leq) = \left\{ \begin{tikzcd}[column sep=small, row sep=small] a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd} \right\}$$

and the two **Ab**-valued presheaves

$$F = \left[\begin{tikzcd}[column sep=small, row sep=small] \mathbb{Z}/(2) \arrow[dr, "0"] & & \mathbb{Z}/(2) \arrow[dl, "1"'] \\ & \mathbb{Z}/(2) & \end{tikzcd} \right] \text{ and } G = \left[\begin{tikzcd}[column sep=small, row sep=small] \mathbb{Z}/(2) \arrow[dr, "1"] & & 0 \arrow[dl] \\ & \mathbb{Z}/(2) & \end{tikzcd} \right].$$

Calculate $\mathrm{Hom}_{\mathrm{presh}_{\mathbf{Ab}}(X,\leq)}(F, G)$ and $\mathrm{Hom}_{\mathrm{presh}_{\mathbf{Ab}}(X,\leq)}(G, F)$.

Exercise 6. Let $\mathtt{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor which is full, faithful, and dense. For $D \in \mathcal{D}$ choose an object $\mathtt{G}D \in \mathcal{C}$ and an isomorphism $\alpha_D\colon \mathtt{FG}D \to D$. (This is possible since $\mathtt{F}$ is dense.) For a morphism $f\colon D_1 \to D_2$ let $\mathtt{G}f$ be the unique morphism such that $\mathtt{FG}f = \alpha_{D_2}^{-1} f \alpha_{D_1}$.

1. Show that $\mathtt{G}$ defines a functor from $\mathcal{D}$ to $\mathcal{C}$.
2. Show that α defines a natural isomorphism from $\mathtt{FG}$ to $\mathrm{id}_{\mathcal{D}}$.
3. Show that $\mathtt{GF}$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$.

Exercise 7. Consider the forgetful functor $\mathtt{forget}\colon \mathbf{Ab} \to \mathbf{Set}$.

1. Show that it has a left adjoint, given by sending a set S to the free abelian group $\mathbb{Z}^{(S)}$.
2. Show that this forget functor does not have any right adjoint functor.

Exercise 8. Consider the forgetful functor $\mathbf{Top} \to \mathbf{Set}$. Construct a right and a left adjoint to this forget-functor.