Exercise 21. For any category $\mathcal{C}$ (not assumed to be additive or abelian), let F be a $\mathcal{C}$ valued presheaf over any poset which has a limit.

Let C be an object in $\mathcal{C}$. Note that if we apply the functor $\mathrm{Hom}(C, -)$ to all the $F(i)$ we obtain a **Set**-valued presheaf. (Equivalently, we obtain this presheaf as the composition of functors $\mathrm{Hom}(C, -) \circ F$.)

Show that

$$\mathrm{Hom}(C, \varprojlim F) = \varprojlim \mathrm{Hom}(C, F(i)).$$

Exercise 22. In an abelian category, consider the diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[r, "f"] & C \\ D \arrow[u] \arrow[r, "g"] & E \arrow[u, "h"'] & \end{tikzcd}$$

where the upper line is exact, and the square is a pullback.

Show that g is an epimorphism if and only if $f \circ h = 0$.

Exercise 23. Let F and G be adjoint functors between two module categories. (With F left adjoint and G right adjoint. You can use that this implies that F is right exact and G is left exact, a fact which I will argue for on Monday.)

Show that F preserves projectives if and only if G is exact.

Exercise 24 (slightly on the side of the main focus of the course — also possibly quite hard)**.** Let R be a ring. Show: A left R-module X is injective if and only if, for each left ideal I of R and any morphism $f\colon I \to X$ there is $x \in X$ such that $f(i) = ix$.

(Hint: For an arbitrary monomorphism, use Zorn's Lemma to show that we have the factorization property of injective objects.)

Conclude that an abelian group is injective if and only if it is "divisible", that is if

$$\forall a \in A \quad \forall n \in \mathbb{N} \setminus \{0\} \quad \exists b \in A\colon a = nb.$$

(Intuitively: any a in the group is divisible by any natural number n.)