Exercise 31. Assume the following map of complexes is a quasi-isomorphism.

$$\begin{tikzcd}
A^\bullet & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots \\
B^\bullet & \cdots \arrow[r] & 0 \arrow[r] & B^{-1} \arrow[r, "f"] & B^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Show that $B^0 = B^{-1} \oplus A^0$, and that f and g correspond to the inclusions of the two summands into the sum.

Exercise 32. Let R be a ring. In the category $\mathbf{C}(\operatorname{Mod} R)$ of complexes of right R-modules, we consider the complex given by putting R itself in degree 0, and 0 in all other degrees, and denote this complex by $R_{\text{in deg } 0}$. That is

$$R_{\text{in deg } 0} = [\cdots \to 0 \to R \to 0 \to \cdots].$$

Show that

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{ModR})}(R_{\text{in deg } 0}, -) \cong \mathrm{H}^0 .$$

Exercise 33. Let $f^\bullet \colon A^\bullet \to B^\bullet$ be a morphism of complexes. Show that the morphism $\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right] : B^\bullet \to \operatorname{Cone}(f)$ is a *weak cokernel* of $f^\bullet$ in the homotopy category.

(That is, the composition $\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right] \circ f^\bullet$ is null-homotopic, and for any map of complexes $g^\bullet \colon B^\bullet \to C^\bullet$ such that $g^\bullet \circ f^\bullet$ is null-homotopic, $g^\bullet$ factors through $\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]$ up to homotopy — the term "weak" refers to the fact that we do not require this factorization to be unique.)