Prop. 22: X unif. convex, Banach $\Rightarrow X$ reflexive

Pf.: Omitted.

For later:

Prop. 23: $X_0 \subset X$ closed lin. subsp.

X reflexive $\Rightarrow X_0$ reflexive.

Pf.: $X_0 \subset X_0''$ always, need $X_0'' \subset X_0$:

1) Take any $x_0'' \in X_0''$, def. $x'' \in X''$ by

$$\langle x'', x' \rangle := \langle x_0'', Rx' \rangle, \quad x' \in X'$$

where $R : X' \to X_0'$ is the restriction.

By Hahn-Banach (Cor. 9), $R(X') = X_0'$

$[\overbrace{x_0' \in X_0'}^{X_0' \subset R(X'):} \Rightarrow \exists\, x' \in X' \text{ w. } x'|_{X_0} = x_0']$ "$R(X') \subset X_0'$ obvious"

2) X'' reflexive $\Rightarrow \exists\, \tilde{x} \in X$ s.t.

$$\langle x'', x' \rangle = \langle x', \tilde{x} \rangle \quad \forall\, x'$$

Obs: $x' \in X_0^\perp \Rightarrow \langle x', x_0 \rangle = 0,\ x_0 \in X_0 \overset{\text{def. } R}{\Rightarrow} Rx' = 0$

$$\Rightarrow \langle x', \tilde{x} \rangle \overset{\text{def } x''}{=} \langle x_0'', \underset{Rx'}{\underset{\parallel}{0}} \rangle \overset{x_0'' \text{ lin}}{=} 0$$

Hence by Thm. 15,

$$\tilde{x} \in \overline{\operatorname{span} X_0} = X_0 \quad \text{(cl. sub sp.)}$$