Examination paper for
MA8109 Stochastic Processes in Engineering Systems

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Examination time (from–to): 15:00–19:00
Permitted examination support material: C: Approved calculator, list of formulas that comes with the exam.

Other information:
There is a list of useful formulas at the end of this exam, read it before you start to work.

Language: English
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Checked by:
Problem 1

Let $B_t$ be a Brownian motion in $\mathbb{R}^1$, show that $\tilde{B}_t = B_{t+1} - B_1$ is another Brownian motion in $\mathbb{R}^1$.

Problem 2

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$ and

$$\phi_N(t, \omega) = \sum_{j=0}^{N} B^2_{t_j}(\omega) \chi_{[t_j, t_{j+1})}(t),$$

where $B_t(\omega)$ is a Brownian motion in $\mathbb{R}^1$ and the function $\chi_A(t)$ is 1 when $t \in A$ and 0 otherwise.

a) Give the definitions of the two Ito integrals

(i) $\int_0^T \phi_N dB_s$,

(ii) $\int_0^T B^2_s dB_s$.

*Hint:* Do not compute these integrals.

b) Find the expectation and variance of

(i) $X_t = \int_0^t s dB_s$,

(ii) $Y = \int_0^T B^2_s dB_s$.

*Hint:* The formulas at the end of the exam may be useful.

Let $\tilde{\phi}$ be the elementary function defined by

$$\tilde{\phi}(t, \omega) = \begin{cases} 
0, & t \in [0, 1), \\
B^2_1(\omega), & t \geq 1,
\end{cases}$$

where $B_1$ is the Brownian motion at $t = 1$.

c) Let $\{\mathcal{F}_t\}_t$ be the filtration generated by the Brownian motion $B_t$ and define

$$X_t(\omega) = \int_0^t \tilde{\phi}(s, \omega) dB_s(\omega).$$

Show by a direct argument that $X_t$ is a martingale w.r.t. the filtration $\{\mathcal{F}_t\}_t$.

*Hint:* The formulas at the end of the exam may be useful.
Problem 3

Physicists have suggested the following (scaled) model for the position of a pollen particle suspended in a liquid that experiences periodic external forcing,

\[ \ddot{X} + \dot{X} = W + \sin t, \]

where \( W \) is Gaussian white noise.

a) Write down the Ito interpretation of this equation.

b) Solve the Ito stochastic differential equation obtained in a).

Hint: Use an integrating factor.

Problem 4

Consider the following stochastic differential equation

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = Z, \] (1)

where \( b \) and \( \sigma \) are continuous functions and \( Z \) is a random variable.

Give sufficient conditions on \( b \), \( \sigma \), and \( Z \) to have existence and uniqueness of a strong solution of (1).

Define what it means for \( X_t \) to be a strong solution of (1).

Problem 5

Let \( f \in C^2_c(\mathbb{R}^2) \) and let \( u \in C^{1,2}(\mathbb{R} \times \mathbb{R}^2) \) be the solution of

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \quad \text{for} \quad (x, y) \in \mathbb{R}^2, \ t > 0, \]

\[ u(0, x, y) = f(x, y) \quad \text{for} \quad (x, y) \in \mathbb{R}^2. \]

Use an Ito diffusion to find an integral formula for \( u(t, x, y) \).

Hint: The formulas at the end of the exam may be useful.
Problem 6  Let $\tau^x_{U_R}$ be the exit-time from $U_R = \{x : |x| < R\}$ of the following one-dimensional Ito diffusion:

$$X_t^x = x + \int_0^t (2 + \sin X_s^x) dB_s.$$

Prove that for all $x \in U_R$ and $R > 0$,

$$E(\tau^x_{U_R}) < \infty.$$

*Hint:* Try the “usual argument”. Formulas at the end of the exam may be usefull.

Problem 7  Let $X_t$ and $X^\epsilon_t$ be (strong) solutions of the one-dimensional (stochastic) differential equations

$$dX_t = b(X_t)dt, \quad X_0 = x_0,$$

$$dX^\epsilon_t = b(X^\epsilon_t)dt + \epsilon dB_t, \quad X^\epsilon_0 = x_0,$$

where $x_0, \epsilon \in \mathbb{R}$, and for all $x, y \in \mathbb{R}$,

$$|b(x) - b(y)| \leq L|x - y|.$$

Prove that there is a $C > 0$ such that

$$E \left( |X_t - X^\epsilon_t|^2 \right) \leq Ce^{Ct}\epsilon^2.$$

*Hint:* The formulas at the end of the exam may be usefull.
List of useful formulae

Note: The list does not state the requirements for the formulae to be valid.

1D Gaussian variable: \( X \in \mathcal{N}(\mu, \sigma^2) \);

(i) \( E(X - \mu)^4 = 3\sigma^4 \),

(ii) \( \Phi_X(u) = e^{i\mu - \frac{1}{2}\sigma^2u^2} \),

(iii) \( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}} \).

Conditional Expectations:

(i) If \( Y \) is \( \mathcal{H} \)-measurable, then \( E(YX|\mathcal{H}) = YE(X|\mathcal{H}) \).

(ii) If \( X \) is independent of \( \mathcal{H} \), then \( E(X|\mathcal{H}) = E(X) \).

(iii) If \( \mathcal{G} \subset \mathcal{H} \), then \( E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{G}) \).

Itô Isometry:

\[
E\left| \int_0^T f(t,\omega) dB_t(\omega) \right|^2 = \int_0^T E |f(t,\omega)|^2 dt = \| f \|^2_{L^2(\Omega \times [0,T])}.
\]

2D Itô Formula: The "Rules" and
\[
dg(t, X_t, Y_t) = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{\partial g}{\partial y} dY_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y} dX_t dY_t + \frac{1}{2} \frac{\partial^2 g}{\partial y^2} (dY_t)^2.
\]

The Generator for \( dX_t = b(X_t) dt + \sigma(X_t) dB_t \):
\[
A(f)(x) = \sum_{i=1}^n b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^n \left( \sigma(x) \sigma(x)^T \right)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x).
\]

Dynkin’s formula: \( Ef(X^x_T) = f(x) + E \left( \int_0^T Af(X^x_s) ds \right) \).

Grönwall’s inequality: If \( v(t) \leq C + A \int_0^t v(s) ds \ldots \), then \( v(t) \leq Ce^{At} \).

Jensen’s inequality: \( \left( \int_0^t v(s) ds \right)^2 \leq t \int_0^t v^2(s) ds \).