

The Jacobson Radical

R a ring, M an R-module

Def: The (Jacobson) radical $\text{Rad } M$ of M is
the intersection of all maximal (proper) submodules of M.

Ex: If M is semisimple (i.e. a possibly infinite direct sum
of simples)

then $\text{Rad } M = 0$

$\Gamma M = \bigoplus_{i \in I} S_i$ for $i \in I : M_i = \bigoplus_{j \in I \setminus \{i\}} S_j$ is a maximal submodule

$$\bigcap_{i \in I} M_i = 0$$

Lemma: $\text{Rad}(M / \text{Rad } M) = 0$

[max submodules of $M / \text{Rad } M \xrightarrow{?=?}$ max submodules of M
containing " " $\text{Rad } M$]

$\text{Rad}(M / \text{Rad } M) = \bigcap_{\substack{\text{max submod} \\ \text{of } M}} / \text{Rad } M = 0$ / all max submodules of M

Rem: same argument shows $N \subseteq \text{Rad } M$ then $\text{Rad}(M/N) = \text{Rad } M/N$

Functionality

$f: M \rightarrow N$ then $f(\text{Rad } M) \subseteq \text{Rad } N$

$\lceil N'$ a max submodule of N

Claim: $f^{-1}(N')$ is either M or a max submodule of M

$$\begin{array}{ccccc} f^{-1}(N') & \hookrightarrow & M & \longrightarrow & M/f^{-1}(N') \\ \downarrow \text{PB} & & \downarrow f & & \downarrow \text{mono} \\ N' & \hookrightarrow & N & \longrightarrow & N/N' \\ & & & & \text{Simple} \end{array}$$

Simple or 0

This proves the claim

Observe that f^{-1} commutes with intersections

$\rightarrow f^{-1}(\text{Rad } N)$ is $\underbrace{\text{an intersection}}_{\text{Max}}$ of max submodules of M

\rightarrow contains intersection of all max submodules of M

$\lceil f^{-1}(\text{Rad } N) \supseteq \text{Rad } M$ i.e. $f(\text{Rad } M) \subseteq \text{Rad } N$

Corollary: $M \cdot \text{Rad } R_R \subseteq \text{Rad } M$

R as a right R -module

$\Gamma_{m \in M} \quad m \cdot - : R_R \longrightarrow M$

$m \text{Rad } R_R \subseteq \text{Rad } M$

Corollary: $\text{Rad } R_R$ is a two-sided ideal

Γ right ideal by construction.

Γ left ideal $R_R \cdot \text{Rad } R_R \subseteq \text{Rad } R_R$ by cor above

Lemma

If M is artinian, then

$$M \text{ semisimple} \Leftrightarrow \text{Rad } M = 0$$

Counterex when
 M not artinian:
 \mathbb{Z} over \mathbb{Z}

[Have seen " \Rightarrow " in general]

So assume $\text{Rad } M = 0$

M artinian $\Rightarrow M$ has a simple submodule S_1

$S_1 \notin \text{Rad } M \rightsquigarrow \exists$ max submodule M_1 of M
s.t. $S_1 \notin M_1$

$$S_1 + M_1 = M \quad (\text{because properly bigger than } M_1)$$

$$S_1 \cap M_1 = 0 \quad (\text{submod of } S_1, \text{ not } S_1)$$

$$M = S_1 \oplus M_1$$

exact same argument for M_1 shows $M_1 = \underbrace{S_2 \oplus M_2}_{\text{simple}}$

$$M = S_1 \oplus M_1 = S_1 \oplus S_2 \oplus M_2 = \dots$$

$M \supseteq M_1 \supseteq M_2 \supseteq M_3 \dots$ has to stop (artinian)

\rightarrow eventually $M_i = 0 \quad M = S_1 \oplus \dots \oplus S_i$

L

Corollary: $M/\text{Rad } M$

$\text{Rad } M$ is the smallest submodule with a semisimple quotient.

$\lceil \text{Rad}(M/\text{Rad } M) = 0 \rightarrow M/\text{Rad } M \text{ is semisimple}$

N any submodule s.t. M/N is semisimple

in particular $\text{Rad}(M/N) = 0$

$$\begin{array}{ccc} M & \xrightarrow{\text{proj}} & M/N \\ \downarrow & & \downarrow \\ \text{Rad}(M) & \longrightarrow & \text{Rad}(M/N) \\ & & \parallel \\ & & 0 \end{array} \quad \text{Rad}(M) \subseteq N$$

\lfloor

Rem: In general: $\text{Rad}(M)$ is contained in any submodule with a semisimple quotient.

Lemma: If R_R artinian, then

$$\text{Rad } M = M \cdot \text{Rad } R_R$$

[Have seen " \subseteq "

for " \subseteq " $R_R/\text{Rad } R_R$ is semisimple

$M/M \cdot \text{Rad } R_R$ is a $R_R/\text{Rad } R_R$ -module

so it is semisimple

$\Rightarrow \text{Rad } M \subseteq M \cdot \text{Rad } R_R$
previous remark

Theorem: R a ring TFAE for $r \in R$

(i) $r \in \text{Rad } R_R$

(ii) $r \in \text{Rad } {}_R R$

(iii) $\nexists s \in R : 1 + rs \text{ invertible}$

(iv) $\nexists s \in R : 1 + sr \text{ invertible}$

(v) $\nexists s, t \in R : 1 + srt \text{ invertible}$

$$m = 1 - rs$$

invertible by (iii)

\Leftrightarrow can't have
invertible in
max ideal

$\rightarrow r \in \text{Rad } R_R$

(i) \Rightarrow (v) dual to (i) \Rightarrow (v)

(iv) \Rightarrow (ii) dual to (iii) \Rightarrow (i)

$\Gamma(v) \Rightarrow (iii)$ and $(v) \Rightarrow (iv)$ are clear

(i) \Rightarrow (v) $r \in \text{Rad } R_R \rightarrow srt \in \text{Rad } R_R$

$\rightarrow srt$ contained in all max right ideals

$\rightarrow 1 + srt$ cannot be contained in any max right ideal

$$\rightarrow (1 + srt)R = R$$

$$\rightarrow \exists h \in R \text{ s.t. } (1 + srt)h = 1$$

$h = 1 - srth \rightarrow$ of the same form \rightarrow also has a
right inverse

$\rightarrow h$ is two-sided inverse to $(1 + srt)$

(iii) \Rightarrow (i) assume $r \notin M$ for some max right ideal M

$$\rightarrow rR + M = R \rightarrow \exists s \in R, m \in M \text{ s.t. } rst + m = 1$$