

$\text{Rad } M = \bigcap \text{ max submodules of } M$

M artinian $\leadsto \text{Rad } M$ smallest submodule s.t. $M/\text{Rad } M$ semisimple

A artinian $\leadsto \text{Rad } M = M \cdot \text{Rad } A$

Fact: M finitely generated, then every proper submodule is contained in a maximal one.

Γ Zorn's lemma: (proper submodules, $\subseteq$) is poset

$\downarrow$ every chain has an upper bound: the union

$\downarrow$ $\exists$ max element

$\uparrow$ proper since M fin gen

Prop: If M is finitely generated, then $\text{Rad } M$ is the biggest superfluous submodule.
= small

(A submodule $N \subseteq M$ is superfluous if $\nexists H \subseteq M$: $N + H = M$ only if $H = M$)
submodule

Γ since every proper submodule is contained in a max one:

$N \subseteq M$ small $\Leftrightarrow \nexists H \subseteq M$
max submodule

$N + H \neq M$

$\Leftrightarrow N \subseteq H$ (since H max)

$\downarrow \Leftrightarrow N \subseteq \text{Rad } M$

Corollary: M fin. gen, then $\text{Rad } M \neq M$ or $M = 0$

Corollary: $P \xrightarrow{f} M$ epi from a ^{fg} projective, then f is a projective com if and only if $\text{Ker } f \subseteq \text{Rad } P$

Corollary: A artinian ring. Then $\text{Rad } A$ is the
biggest $\cdot$ nilpotent ideal
 $\cdot$ nil left/right ideal

Γ A artinian $\leadsto$ all right ideals fin gen.

$A \neq \text{Rad } A \neq (\text{Rad } A)^2 \neq \dots$

needs to stop
since A artinian

$\leadsto \exists n$ s.t. $(\text{Rad } A)^n = 0$.

it is a nil right ideal

$$\forall n \in \mathbb{N}: \forall a \in A \quad 1 + \underbrace{na}_{\in N} \text{ invertible} \\ 1 - na + (na)^2 - (na)^3 + \dots$$

$$L \leadsto n \in \text{Rad } A$$

Examples

$$\mathbb{Z}_2 \quad \text{Rad } \mathbb{Z}_2 = 0$$

$$k[X] \quad \text{Rad } k[X] = 0$$

$$(X-a)$$

is maximal for $a \in k$

$$k[[X]] \quad \text{Rad } k[[X]] = (X)$$

everything with a non-zero constant term is invertible

$$kQ \quad \text{Rad } kQ = (\text{the arrows})$$

Q a finite acyclic quiver

$$M_n(k) \quad n \times n \text{ - matrices over } k$$

$$\text{Rad } M_n(k) = 0$$

$M_n(k)$ has no non-trivial two-sided ideals

$$T_n(k) \quad \text{upper triangular } n \times n \text{ - matrices over } k$$

$$\text{Rad } T_n(k) = \{\text{matrices with zero diagonal}\}$$

$$k[X]/(X^2 - X) \quad \text{Rad } k[X]/(X^2 - X) = 0$$

$$\{a + bx\} \quad \text{when is } a + bx \text{ nilpotent?}$$

$$(a + bx)^2 = a^2 + 2abx + b^2x \\ \times \quad x^2 = x$$

$$a = 0 \\ b = 0$$

$$k[X]/(X^2 - X) \cong k[X]/(X) \times k[X]/(X-1) \cong k \times k$$

$$X^2 - X = X(X-1)$$

Projectives and simples

A artinian ring

indecomposable = cannot be written as direct sum of two submodules

Prop: The functor $P \mapsto P/\text{Rad } P$ gives bijections

$$\text{indec. proj} / \cong \xrightarrow{1:1} \text{simples} / \cong$$

$$\text{f.g. proj} / \cong \xrightarrow{1:1} \text{f.g. semisimples} / \cong$$

Γ know: $P/\text{Rad } P$ is semisimple

construction in the other direction: projective cover
(note: this is not a functor)

know: P is the projective cover of $P/\text{Rad } P$

S simple, $P \twoheadrightarrow S$ a proj. cov., i.e.
 $\text{Ker } p \subseteq \text{Rad } P$
?
max, because simple quotient

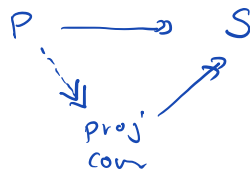
$$\Rightarrow \text{Ker } p = \text{Rad } P$$

note: this also means that P cannot decompose

$$\text{(otherwise } P = P_1 \oplus P_2, \text{ Rad } P = \text{Rad } P_1 \oplus \text{Rad } P_2 \\ P/\text{Rad } P = \underbrace{P_1/\text{Rad } P_1}_{\neq 0} \oplus \underbrace{P_2/\text{Rad } P_2}_{\neq 0})$$

If P indec proj: assume $P/\text{Rad } P$ not simple

$$\exists P/\text{Rad } P \twoheadrightarrow S \text{ simple}$$



$\leadsto$ proj. cov. is a summand of P

$$\leadsto \text{proj. cov.} = P$$

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