$\mathrm{Rad}\, M = \bigcap$ max submodules of M

M artinian $\rightsquigarrow$ $\mathrm{Rad}\, M$ smallest submodule s.t. $M/\mathrm{Rad}\, M$ semisimple

A artinian $\rightsquigarrow$ $\mathrm{Rad}\, M = M \cdot \mathrm{Rad}\, A$

Fact: M finitely generated, then every proper submodule is contained in a maximal one.

Zorn's lemma: (proper submodules, $\subseteq$) is poset

every chain has an upper bound: the union

proper since M fin gen

$\Downarrow$

$\exists$ max element

Prop: If M is finitely generated, then $\mathrm{Rad}\, M$ is the biggest superfluous submodule.

= small

(A submodule $N \subseteq M$ is superfluous if $\forall H \subseteq M$ submodule: $N + H = M$ only if $H = M$)

Since every proper submodule is contained in a max one:

$$N \subseteq M \text{ small} \iff \forall \underset{\text{max submod}}{H \subseteq M} \quad \underbrace{N + H \neq M}_{\iff N \subseteq H \text{ (since } H \text{ max)}}$$

$$\iff N \subseteq \mathrm{Rad}\, M$$

Corollary: M fin. gen, then $\mathrm{Rad}\, M \subsetneq M$ or $M = 0$

Corollary: $P \xrightarrow{f} M$ epi from a f.g. projective, then f is a projective cover if and only if $\mathrm{Ker}\, f \subseteq \mathrm{Rad}\, P$

Corollary: A artinian ring. Then $\mathrm{Rad}\, A$ is the biggest
- nilpotent ideal
- nil left / right ideal

A artinian $\rightsquigarrow$ all right ideals fin gen.

$A \supsetneq \mathrm{Rad}\, A \supsetneq (\mathrm{Rad}\, A)^2 \supsetneq, \ldots$ needs to stop since A artinian

$\rightsquigarrow \exists n$ s.t. $(\mathrm{Rad}\, A)^n = 0$.

if N is a nil right ideal

$\forall n \in N: \forall a \in A \quad 1 + \underbrace{na}_{\in N}$ invertible

$$1 - na + (na)^2 - (na)^3 + \dots$$

$\rightsquigarrow n \in \operatorname{Rad} A$

<u>Examples</u>

$\mathbb{Z}_{\mathbb{Z}}$ $\quad \operatorname{Rad} \mathbb{Z}_{\mathbb{Z}} = 0$

$k[X]$ $\quad \operatorname{Rad} k[X] = 0$ $\quad (X-a)$ is maximal for $a \in k$

$k[[X]]$ $\quad \operatorname{Rad} k[[X]] = (X)$ $\quad$ everything with a non-zero constant term is invertible

kQ $\quad \operatorname{Rad} kQ = (\text{the arrows})$

Q a finite acyclic quiver

$M_n(k)$ $\quad n \times n$ - matrices over k

$\operatorname{Rad} M_n(k) = 0$ $\quad M_n(k)$ has no non-trivial two-sided ideals

$T_n(k)$ $\quad$ upper triangular $n \times n$ - matrices over k

$\operatorname{Rad} T_n(k) = \{\text{matrices with zero diagonal}\}$

$k[x]/(x^2 - x)$ $\quad \operatorname{Rad} k[x]/(x^2-x) = 0$

$\quad \shortparallel$

$\{a + bx\}$ $\quad$ when is $a+bx$ nilpotent?

$$(a+bx)^2 = a^2 + 2abx + b^2x \qquad a = 0$$

$$x \qquad x^n = x \qquad b = 0$$

$$k[x]/(x^2 - x) \cong k[x]/(x) \times k[x]/(x-1) \cong k \times k$$

$$x^2 - x = x(x-1)$$

Projectives and simples

A artinian ring

indecomposable = cannot be written as direct sum of two submodules

Prop: The functor $P \longmapsto P/\mathrm{Rad}\, P$ gives bijections

$$\text{indec. proj}/\cong \;\xrightarrow{1:1}\; \text{simples}/\cong$$

$$\text{f.g. proj}/\cong \;\xrightarrow{1:1}\; \text{f.g. semisimples}/\cong$$

know: $P/\mathrm{Rad}\, P$ is semisimpel

construction in the other direction: projective cover
(note: this is not a functor)

know: P is the projective cover of $P/\mathrm{Rad}\, P$

S simpel, $P \xrightarrow{p} S$ a proj cover, i.e. $\ker p \subseteq \mathrm{Rad}\, P$

$\uparrow$ max, because simpel quotient

$\Rightarrow \ker p = \mathrm{Rad}\, P$

note: this also means that P cannot decompose
(otherwise $P = \underset{\neq 0}{P_1} \oplus \underset{\neq 0}{P_2}$, $\mathrm{Rad}\, P = \mathrm{Rad}\, P_1 \oplus \mathrm{Rad}\, P_2$

$P/\mathrm{Rad}\, P = \underbrace{P_1/\mathrm{Rad}\, P_1}_{\neq 0} \oplus \underbrace{P_2/\mathrm{Rad}\, P_2}_{\neq 0}$)

If P indec proj: assume $P/\mathrm{Rad}\, P$ not simpel

$\exists\; P/\mathrm{Rad}\, P \twoheadrightarrow S$ simpel

$$\begin{tikzcd}
P \arrow[rr, two heads] \arrow[dr, dashed] & & S \\
& \text{proj cov} \arrow[ur, two heads] &
\end{tikzcd}$$

$\rightarrow$ proj cov is a summand of P

$\rightarrow$ proj cov $= P$