(A artinian)

Prop $P \mapsto P/\mathrm{Rad}\,P$ gives bijections
indec proj $/\cong$ $\longrightarrow$ simples $/\cong$ and f.g. proj $/\cong$ $\longrightarrow$ f.g. semisimples $/\cong$

Cor: P indec proj $\Rightarrow$ $\mathrm{Rad}\,P$ is a maximal submodule
Cor: A_A indec $\Leftrightarrow$ $\mathrm{Rad}\,A$ is a maximal right ideal
$\overset{\text{def}}{\Leftrightarrow}$ A <u>local</u>.

Observation: A local $\Leftrightarrow$ $\forall$ non-invertible $a \in A$: $1 + a$ invertible
($\Leftrightarrow$ $\{\text{non-invertibles}\} = \mathrm{Rad}\,A$)

Thm (Krull-Schmidt)
M a finite dimensional module over some k-algebra
Then M has an <u>essentially unique</u> decomposition into indecomposables,
unique up to isomorphism and permutation
↳ cannot be written as direct sum (except self $\oplus$ 0)

Moreover these indecomposables all have local endomorphism rings.

Rem: existence of a decomposition into indec. is clear: M has finite dimension, for each decomp. step dimension goes down

Lemma 1: M a f.d. module, then
M indec $\Leftrightarrow$ $\mathrm{End}(M)$ local

$\ulcorner$ "$\Leftarrow$" (true in general) $M = M_1 \oplus M_2$ $\quad M_1 \underset{\iota_1}{\overset{\pi_1}{\leftrightarrows}} M \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} M_2$

$\mathrm{id}_M = \iota_1\pi_1 + \iota_2\pi_2$ in $\mathrm{End}(M)$

one of these needs to be invertible by "local"

$\rightsquigarrow$ corresponding summand is $= M$

" $\Rightarrow$ " note $\mathrm{End}_A(M)$ is a finite dimensional algebra

$$(\subseteq \mathrm{End}_k(M) \cong M_{\dim M}(k))$$

$\rightsquigarrow$ artinian

If $\mathrm{End}_A(M)$ not local $\Rightarrow$ $\mathrm{End}_A(M) = \overset{\neq 0}{P} \oplus \overset{\neq 0}{Q}$

as right $\mathrm{End}_A(M)$-modules

$M = \mathrm{End}_A(M) \otimes_{\mathrm{End}_A(M)} M$ (note: M is naturally an $\mathrm{End}(M)-A-$bimodule)

$= P \otimes_{\mathrm{End}_A(M)} M \oplus Q \otimes_{\mathrm{End}_A(M)} M$

since M is indec: one of them is 0, say $P \otimes_{\mathrm{End}(M)} M = 0$.

$$P \otimes_{\mathrm{End}_A(M)} M \xrightarrow{\mathrm{incl} \otimes \mathrm{id}} \mathrm{End}_A(M) \otimes_{\mathrm{End}_A(M)} M \xrightarrow{\cong} M$$

$$\varphi \otimes m \longmapsto \varphi \otimes m \longmapsto \varphi(m)$$

$\varphi \in P$, $m \in M$ $\qquad \varphi \neq 0 \rightsquigarrow \exists m$ s.t. $\varphi(m) \neq 0$

$\rightsquigarrow \varphi \otimes m \neq 0$ as element in $P \otimes_{\mathrm{End}_A(M)} M$ $\lightning$

$\varphi = 0$ is only element of P $\lightning$ $P \neq 0$

$\llcorner$

<u>Rem</u>: Statement is also true for M finite length (but needs a new proof)

<u>Lemma 2</u>: $M = \bigoplus_{i=1}^m M_i$ $\qquad M_i$ has local endo-ring

$M = \bigoplus_{i=1}^n N_i$ $\qquad N_i$ indec

Then the M_i's and N_i's coincide up to iso and permutation.

$\ulcorner$ Induction on m. $\quad m = 1$ ✓

$$M = M_1 \oplus M_{rest} \underset{\begin{bmatrix} \alpha_1 & \\ \vdots & * \\ \alpha_n & \end{bmatrix}}{\overset{\begin{bmatrix} \beta_1 \cdots \beta_n \\ * \end{bmatrix}}{\rightleftarrows}} \bigoplus_{i=1}^n N_i$$

composition is id_M

$\rightsquigarrow \mathrm{id}_{M_1} = \sum_{i=1}^n \beta_i \alpha_i$

$\mathrm{End}(M_1)$ local $\Rightarrow \exists i$ s.t. $\beta_i \alpha_i$ invertible

wlog $\beta_1 \alpha_1$ invertible

$\rightsquigarrow M_1$ is iso to a direct summand of N_1

$\xrightarrow{N_1 \text{ indec}}$ $M_1 \cong N_1$, β_1, α_1 isos

$$\begin{tikzcd}[column sep=huge, row sep=large]
M_1 \oplus M_{rest} \arrow[r, "\cong", "{\begin{bmatrix} \alpha_1 & \gamma \\ \alpha_{rest} & * \end{bmatrix}}"'] & N_1 \oplus N_{rest} \arrow[d, "\cong"', "{\begin{bmatrix} 1 & 0 \\ -\alpha_{rest}\alpha_1^{-1} & 1 \end{bmatrix}}"] \\
M_1 \oplus M_{rest} \arrow[u, "{\begin{bmatrix} 1 & -\alpha_1^{-1}\gamma \\ 0 & 1 \end{bmatrix}}", "\cong"'] \arrow[r, "{\begin{bmatrix} \alpha_1 & 0 \\ 0 & ** \end{bmatrix}}"', "\text{from matrix mult.}"] & N_1 \oplus N_{rest}
\end{tikzcd}$$

iso $\Leftrightarrow$ α_1 and $**$ isos

$\rightsquigarrow M_{rest} \cong N_{rest}$ $\rightarrow$ inductively: summands in M_{rest} are same as in N_{rest} up to iso and ordering

$\llcorner$

Now Krull-Schmidt theorem follows from Lemma 1 and Lemma 2.

General setup for this course:

A a finite dimensional algebra

want to understand mod A (modules = representations)

Krull-Schmidt theorem: may focus on indecomposable modules (and morphisms between indecomposable modules)

<u>Exercises</u> Find all indec. (contravariant) representations of

$$Q = \begin{tikzcd} 0 \arrow[d] \\ 1 \end{tikzcd} \qquad\qquad Q = \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd}$$

$$Q = \begin{tikzcd} & 0 \arrow[dl] \arrow[d] \arrow[dr] & \\ 1 & 2 & 3 \end{tikzcd}$$

Show that $Q = \begin{tikzcd} & 0 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ 1 & 2 & 3 & 4 \end{tikzcd}$ has ∞-many indec. representations.

Morita equivalence

Theorem: R, S rings

$\mathrm{mod}\, R \approx \mathrm{mod}\, S \iff \exists\, P \in \mathrm{mod}\, R$ a progenerator

- projective
- $\forall M \in \mathrm{mod}\, R$ $\exists$ epi $P^n \twoheadrightarrow M$

s.t. $S = \mathrm{End}(P)$

Γ "$\Rightarrow$" $\mathrm{mod}\, R \approx \mathrm{mod}\, S$

$P \longleftrightarrow S_S$ is a progenerator

$\Leftarrow$ P also progenerator

$\mathrm{End}_S(S) = S$

$\Rightarrow \mathrm{End}_R(P) = S$

"$\Leftarrow$" note: P is an S-R-bimodule

$$\mathrm{mod}\, R \underset{\mathrm{Hom}_R(P,-)}{\overset{-\otimes_S P}{\rightleftarrows}} \mathrm{mod}\, S$$

adjoint pair of functors

both are right exact (P is projective)

$$\begin{array}{ccc} \cup & & \cup \\ \{P^n\} & \overset{\approx}{\rightleftarrows} & \{S^n\} \end{array}$$

this equivalence propagates to cokernels of maps between P^n's / S^n's — these are all of $\mathrm{mod}\, R$ / $\mathrm{mod}\, S$

$\llcorner$