Torsdag 31. mars

Q Euclidean, f additive function

M indecomposable

M preproj. $\Leftrightarrow$ $\Phi^n \underline{\dim} M = \underline{\dim} M + \alpha f$, $\alpha < 0$

M regular $\Leftrightarrow$ $\alpha = 0$

M preinj. $\Leftrightarrow$ $\alpha > 0$

Prop: Q connected Euclidean

Then $\mathcal{R} := \operatorname{add}(\text{regular modules})$ is an abelian subcategory

arbitrary finite sums

of $\operatorname{mod} kQ$ (i.e. the inclusion $\mathcal{R} \hookrightarrow \operatorname{mod} kQ$ is exact)

Pf:

$M, N \in \mathcal{R}$, $f: M \to N$

Every indec summand of $\operatorname{Im} f$ has a non-zero map to some indec. summand of N $\rightsquigarrow$ this summand is not pre inj.

dual argument: no summand of $\operatorname{Im} f$ is preproj

$\Rightarrow \operatorname{Im} f \in \mathcal{R}$

Since summands of $\operatorname{Ker} f$ have maps to M, $\operatorname{Ker} f$ consists of preproj & regular summands

For n as in the previous proposition:

$$\Phi^n \underline{\dim} \operatorname{Ker} f = \Phi(\underline{\dim} M - \underline{\dim} \operatorname{Im} f)$$
$$= \underline{\dim} M - \underline{\dim} \operatorname{Im} f$$
$$= \underline{\dim} \operatorname{Ker} f$$

Since no preinjective summands: this implies all summands of $\operatorname{Ker} f$ are regular.

Dually all summands of $\operatorname{Cok} f$ are regular $\square$

Notation: When using concepts for abelian categories in $\mathcal{R}$ we use "quasi-"

quasi-simples = simples in $\mathcal{R}$

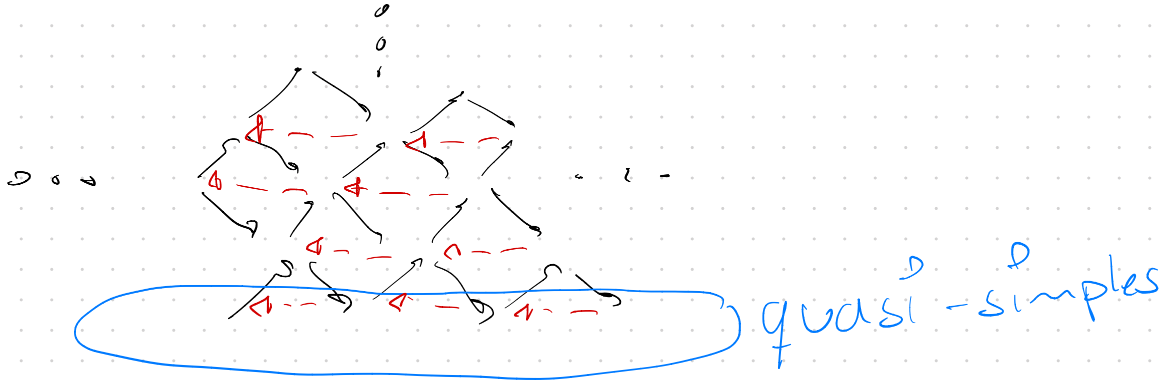
(note: simple KQ-modules that lie in $\mathcal{R}$ are quasi-simples, but there will be more quasi-simples.)

Note:

τ is an auto-equivalence on $\mathcal{R}$ (since in $\mathcal{R}$ nothing factors through projectives or injectives)

Note:

Quasi-simple modules need to be at the lower edge of the AR-components $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\tau^n$



If M is quasi-simple, then its entire τ-orbit consists of quasi-simples.

Lemma: If M is quasi-simple then $\exists\, n>0$ s.t. $\tau^n M = M$.

Pf: Know: $\exists\, n>0$ s.t. $\underbrace{\Phi^n(\underline{\dim} M)}_{\underline{\dim}\, \tau^n M} = \underline{\dim} M$

Pick n minimal with this property

Recall: $(\cdot, \cdot)$ is positive semi-definite

- If $(\underbrace{\underline{\dim} M}_{=\tau^{n}\underline{\dim} M}, \underline{\dim} M) > 0$ then at least one of $\mathrm{Hom}(\tau^{n} M, M)$ or $\mathrm{Hom}(M, \tau^{n} M)$ is none-zero. Both are quasi-simple, so any such non-zero map will be an iso.
- If $(\underline{\dim} M, \underline{\dim} M) = 0$ then $\underline{\dim} M$ is an additive function

$\Gamma(\circ,\circ)$ pos. def on $\mathbb{Z}^{\#\text{vertices}} / (f)$

$\rightsquigarrow$ $\Phi\, \underline{\dim} M = \underline{\dim} M$ so $n=1$

<u>note:</u> $\mathrm{Ext}^1(M, \tau M) \neq 0$ contains almost split sequences

$(\underline{\dim} M, \underline{\dim}\, \tau M) = 0 \Rightarrow$ at least one of

$\mathrm{Hom}(M, \tau M)$ or $\mathrm{Hom}(\tau M, M)$ is non-zero

as before, this means that $M \cong \tau M$

M quasi-simple

$$\begin{tikzcd}
{} & M^{(3)} \arrow[l] \arrow[dr, two heads] & & \tau^{-}M^{(3)} \arrow[ll] \arrow[dr, two heads] & & \tau^{-2}M^{(3)} \arrow[ll] \\
{} & & M^{(2)} \arrow[ll] \arrow[ur, hook] \arrow[dr, two heads] & & \tau^{-}M^{(2)} \arrow[ll] \arrow[ur, hook] \arrow[dr, two heads] & & {} \arrow[ll] \\
& \tau M \arrow[ur, hook] & & M \arrow[ll] \arrow[ur, hook] & & \tau^{-}M \arrow[ll]
\end{tikzcd}$$

Lemma

Any morphism from an indec to M not starting in

$M, M^{(2)}, \dots, M^{(n-1)}$

factors through

$$M^{(n)} \twoheadrightarrow M^{(n-1)} \twoheadrightarrow \dots \twoheadrightarrow M$$

Pf: By induction on n

$n=2$: $\tau M \hookrightarrow M^{(2)} \twoheadrightarrow M$ is almost split

$\rightsquigarrow$ every map except autos of M factors through factors.

Assume ok for $n-1$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr, bend left] \arrow[rr, "\exists \text{ inductively}"] & & M \\
M^{(n)} \oplus \tau M^{(n-2)} \arrow[rr, two heads] & & M^{(n-1)} \arrow[ur] &
\end{tikzcd}$$

not iso by assumption $(X \not\cong M^{(n-1)})$

<u>note</u>: $\tau M^{(n-2)} \hookrightarrow M^{(n-1)} \twoheadrightarrow M$ composes to zero

$$
\left[
\begin{tikzcd}
M^{(n-1)} \arrow[r, two heads] & M^{(n-2)} \arrow[r, two heads] & M^{(n-3)} \arrow[r, two heads] & \cdots \arrow[r, two heads] & M^{(2)} \arrow[r, two heads] & M \\
\tau M^{(n-2)} \arrow[u, hook] \arrow[r, two heads] & \tau M^{(n-3)} \arrow[u, hook] \arrow[r, two heads] & \tau M^{(n-4)} \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \tau M \arrow[u, hook] \arrow[ur, "0"'] &
\end{tikzcd}
\right]
$$

$\rightsquigarrow$ can forget about component $X \to \tau M^{(n-2)}$
so our map $X \to M$ factors through $M^{(n)} \twoheadrightarrow M$

$\square$

Lemma

$M^{(n)}$ is quasi-uniserial

Pf: Enough to show that $M^{(n)} \twoheadrightarrow M$ is the unique map from $M^{(n)}$ to a quasi-simple (Then unique maximal submodule $\tau M^{(n-1)}$ induction)

Note: $M^{(n)}$ has quasi-length n

It follows that $\tau M^{(n-1)} \hookrightarrow M^{(n)} \twoheadrightarrow M$ is exact

Proof by induction on n.

$n=1$. Ok

$$\begin{tikzcd} M^{(n)} \arrow[r, two heads] & N \\ \tau M^{(n-1)} \arrow[u, hook] \arrow[ur, "*"'] & \end{tikzcd}$$

quasi-simple

if the composition $*$ vanishes then $M^{(n)} \twoheadrightarrow N$ is $M^{(n)} \twoheadrightarrow M$

otherwise, inductively: $\tau M^{(n-1)} \hookrightarrow \tau M$

$$\begin{tikzcd}
\tau M^{(n-1)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\wr q"] & M^{(n)} \arrow[r, two heads] \arrow[d] \arrow[dl] & M \arrow[d, equal] \\
\tau M \arrow[r, hook] & M^{(2)} \arrow[r, two heads] & M
\end{tikzcd}$$

$\leftarrow$ non-split

should split by factorization. $\lightning$

$\square$

<u>Proposition</u>

Any indec regular module over KQ, Q Euclidean, is of the form $M^{(n)}$ for M quasi-simple

<u>Pf:</u>

Let X be regular indec

$n =$ quasi-length(X)

Let M ~~be~~ quasi-simple s.t. $\mathrm{Hom}(X, M) \neq 0$

Know: X is not iso to $M, M^{(1)}, \ldots, M^{(n-1)}$
(has different quasi-lengths)

$$\rightsquigarrow \exists \quad \begin{tikzcd} X \arrow[r, two heads] \arrow[dr, "\varphi"'] & M \\ & M^{(n)} \arrow[u, "f"'] \end{tikzcd}$$

If φ is not epi, then φ factors through $\tau M^{(n-1)} \hookrightarrow M^{(n)}$
(this is the unique maximal subobject)

↯ composition
$\tau M^{(n-1)} \hookrightarrow M^{(n)} \twoheadrightarrow M$ is zero

so φ is epi.

Since quasi-length(X) = quasi-length$(M^{(n)})$ it follows that φ is zero. ∎

Theorem

Q connected Euclidean. Then the category of regular kQ-modules is uniserial (all indecs are uniserial) and its AR-quiver consists of components on the form $\mathbb{Z}A_\infty/(\tau^{n_i})$ for certain n_i.

There are no maps between different regular components.