<u>Theorem</u> R ring TFAE for $r \in R$

(i) $r \in \operatorname{rad} R_R$
(ii) $r \in \operatorname{rad}_R R$
(iii) $\forall s \in R \mid 1 + rs$ is invertible
(iv) $\forall s \in R \mid 1 + sr$ is invertible
(v) $\forall s, t \in R \mid 1 + srt$ is invertible

<u>Pf.</u> (v) $\Rightarrow$ (iii) & (iv) ✓

(i) $\Rightarrow$ (v) $r \in \operatorname{rad} R_R \Rightarrow srt \in \operatorname{rad} R_R$
$\Rightarrow$ srt is contained in every maximal ideals.
$\Rightarrow$ $1 + srt$ cannot be contained in any maximal ideal.
$\Rightarrow (1+srt)R = R \Rightarrow \exists h \in R$ s.t. $(1+srt)h = 1$
$\Rightarrow h = 1 - srth$ is of the same form and has a right inverse. $\Rightarrow h$ is 2-sided inverse to $1 + srt$

(iii) $\Rightarrow$ (i) Assume $r \notin M$ for some maximal M
$\Rightarrow rR + M = R \Rightarrow \exists s \in R, m \in M$ s.t. $rs + m = 1$
$\Rightarrow m = 1 - rs$ is invertible by assumption.
but $m \in M$ ↯ so $r \in M. \Rightarrow r \in \operatorname{rad} M$ □

Lemma

If M is artinian, then:

M is semi-simple $\Leftrightarrow$ $\operatorname{rad} M = 0$

Pf: M artinian $\Rightarrow$ M has a simple submodule S_1

Assume $\operatorname{rad} M = 0$, then $S_1 \not\subseteq \operatorname{rad} M$

$\rightsquigarrow$ $\exists$ max submodule M_1 of M st. $S_1 \not\subseteq M_1$.

$S_1 + M_1 = M$ and $S_1 \cap M_1 = 0$ (submod of S_1 which is not S_1)

$\Rightarrow M \simeq S_1 \oplus M_1$.

Induction on M_1 to get $M_1 \simeq S_2 \oplus M_2$

$\Rightarrow M \simeq S_1 \oplus S_2 \oplus \dots \simeq S_1 \oplus \dots \oplus S_n$ as

$M \supset M_1 \supset M_2 \supset \dots$ is a finite chain by artinian, so M_n is simple. $\square$

Corollary

M artinian

$\operatorname{rad} M$ is the smallest submodule with a semi-simple quotient.

Pf: $\operatorname{rad}(M/\operatorname{rad} M) = 0 \Leftrightarrow M/\operatorname{rad} M$ is s.s.

Let N be any submod s.t. M/N is s.s. and then $\operatorname{rad}(M/N) = 0$.

$$\begin{tikzcd} M \arrow[r] & M/N \\ \operatorname{rad} M \arrow[u, hook] \arrow[r] \arrow[ur, "0"] & \operatorname{rad} M/N = 0 \arrow[u, hook] \end{tikzcd} \Rightarrow \operatorname{rad} M \subseteq N$$

$\square$

Remark In general $\operatorname{rad} M$ is contained in any submodule with s.s. quotient.

Semi-simple is now denoted as s.s.

Lemma

R_R artinian

$\operatorname{rad} M = M \operatorname{rad} R_R$

Pf: Have seen that $\operatorname{rad} M \supseteq M \operatorname{rad} R_R$.

We know that $R_R / \operatorname{rad} R_R$ is s.s. so $M / M \operatorname{rad} R_R$ is a module over $R_R / \operatorname{rad} R_R$

so $M / M \operatorname{rad} R_R$ is s.s.

Then $M \operatorname{rad} R_R$ has s.s. quotient $\Rightarrow \operatorname{rad} M \subseteq M \operatorname{rad} R_R$ $\square$

Rem: same argument shows that if $N \subseteq \mathrm{rad}\, M$
$\Rightarrow$ $\mathrm{rad}(M/N) = \mathrm{rad}(M)/N$.

Functoriality $f: M \to N$ then $\mathrm{rad}\, f: \mathrm{rad}\, M \to \mathrm{rad}\, N$.

Let N' be a max submodule of N.
claim: $f^{-1}(N') = M$ or a maximal submod of M.

$$\begin{tikzcd} f^{-1}(N') \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M \arrow[r, two heads] \arrow[d, "f"'] & M/f^{-1}(N') \arrow[d, tail, "\text{mono}"] \\ N' \arrow[r, hook] & N \arrow[r, two heads] & N/N' \end{tikzcd}$$

$\Rightarrow M/f^{-1}(N')$ is simple or 0.

Observe that f^{-1} commutes with $\cap$ (intersection)

$\rightsquigarrow$ $f^{-1}(\mathrm{rad}\, N)$ is an intersection of maximal submodules of M.
$\Rightarrow$ containes intersection of all maximal submodules of M

$f^{-1}(\mathrm{rad}\, N) \supseteq \mathrm{rad}\, M \Rightarrow \mathrm{rad}\, N \supseteq f(\mathrm{rad}\, M)$ $\square$

Remark If M has no maximal submodules, then $\mathrm{rad}(M) = M$ is definition. This gives us functoriality.

Corollary $M \cdot \mathrm{rad}\, R_R \subseteq \mathrm{rad}\, M$

Pf: pick $m \in M$, then $m \cdot \_ : R_R \to M$. Thus
$m\, \mathrm{rad}\, R_R \subseteq \mathrm{rad}\, M \quad \forall m \in M \Rightarrow M\, \mathrm{rad}\, R_R \subseteq \mathrm{rad}\, M$. $\square$

Corollary $\mathrm{rad}\, R_R$ is a two-sided ideal.

Pf: Right ideal by construction. Notice that $\forall r \in R_R$,
$r\, \mathrm{rad}\, R_R \subseteq \mathrm{rad}\, R_R \Rightarrow$ Left idealness $\square$

Modules:

- In this course the modules will be right modules
- A module over a k-algebra is a k-vector space.

mod^A is the category of finitely presented right A-modules

Mod^A is the category of all right modules

Left modules are denoted as $\mathrm{mod}A$ or $\mathrm{Mod}A$.

Prop: Let A be a finite dim. k-algebra, then finitely presented is the same as finitely generated is the same as finite dimensional.

Remark: Holds whenever A is noetherian

With our conventions:

$\mathrm{Mod}^{k\Gamma}$ = contravariant representations of Γ i.e. a vector space for every vertex and a linear map in opposite direction for every arrow.

The Jacobsen radical

Let R be a ring and M an R-module.

Def. Jacobsen radical
$\mathrm{rad}\, M$ is the intersection of all maximal submodules of M.

Ex If M is semisimple, then $\mathrm{rad}\, M = 0$

Lemma The $\mathrm{rad}(M/\mathrm{rad}\, M) = 0$

Pf: The maximal submodules of $M/\mathrm{rad}\, M$
$\underset{1-1}{\leftrightarrow}$ To maximal submodules of M containing $\mathrm{rad}\, M$.

Then $\mathrm{rad}\, M = \mathrm{rad}^2 M = 0$ in $M/\mathrm{rad}\, M$. □

Representation theory of algebras

Week 1.

Notation:

- k field

Def:

A $\overbrace{\text{}}^{\text{unital}}$ k-algebra is a functor $\Delta \to \mathrm{Mod}\,k$

Examples:

- k • $k[x]$ • $k[x, \dots]$: $k\langle x, \dots \rangle$ ← non-commutative
- $k\Gamma$, where Γ is a quiver
- kG, where G is a group
- $k \times k$ • $M_n(k)$ $n \times n$ matrices

Quiver algebras:

A quiver is a set of vertices and a set of arrows together with functions h (head), t (tail) from arrows to vertices.
I.e. it is a directed graph with no restrictions.

Ex:

- $1 \xrightarrow{a} 2$
- $\begin{tikzcd} 1 \arrow[loop above, "b"] \arrow[loop left, "c"] \arrow[loop right, "a"] \end{tikzcd}$
- $\begin{tikzcd} 1 \arrow[r, bend left=50, "b"] \arrow[r, bend left=20, "a"'] & 2 \arrow[l, bend left=40, "c"] \end{tikzcd}$

A quiver is finite if both sets are finite

A path in a quiver is a sequence of arrows s.t. every arrow start where the previous arrow ended, i.e. $h(a_n) = t(a_{n-1})$.

Def:

Let Q be a quiver, kQ is the free v.s. generated by the paths. It gains an algebra structure by composing paths.

Ex:

$1 \xrightarrow{a} 2 \xrightarrow{b} 3$ paths: $a, b, ba, \overbrace{e_1, e_2, e_3}^{\text{lazy paths}}$