

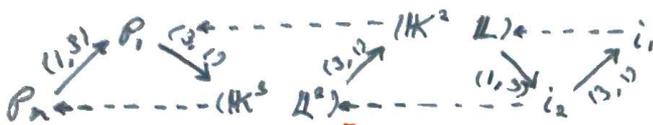
# Reptori - Week 10

## Exercises

1.  $K \subseteq L$  where  $[L:K] = 3$  are fields. Find the AR-quiver of the algebra:

$$\begin{pmatrix} K & L \\ 0 & L \end{pmatrix} = A.$$

We know that  $A \cong (K \ L) \oplus (0 \ L)$ .  
Moreover  $\text{rad } P_1 = P_2$ .



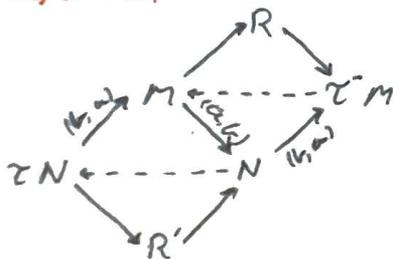
The injectives are  $i_1 = (K \ 0)$  and  $i_2 = (L \ L)$ .

Calculating  $\tau^{-1}$

$$\tau^{-1} P_2 : P_2 \rightarrow i_2 \rightarrow i_1^3$$

$$P_2 \rightarrow P_1^3 \rightarrow (K^3 \ L^2)$$

2. Assume there is an arrow  $M \xrightarrow{(a,b)} N$  in the AR-quiver, s.t.  $a, b \geq 2$ . To this end we may assume  $a, b = 2$ .



$$\dim M + \dim \tau M = b \dim N + \dim R \geq 2 \dim N$$

$$\dim N + \dim \tau N = a \dim M + \dim R' \geq 2 \dim M$$

$\dim M \neq \dim N$  irr. mod. is either mono or epi.

Assume  $\dim M < \dim N \Rightarrow \dim \tau^{-1} M > \dim N$ .

Use this inductively to get that  $\dim(\tau^{-1})^n M > \dim(\tau^{-1})^{n-1} M$  for any  $n$ .

Analogously if  $\dim N < \dim M$ .

$M$  is not injective

dimension vector  
of the projectives

3. Find an algebra  $A$  s.t.  $([P_i])_{i=1}^n$  is a basis for  $K_0(\text{mod } A)$ .

We know that  $\text{gl. dim } A = \infty$ , otherwise we find every module as an alternating sum of its projective resolution.

Fact

Every group algebra s.t.  $|K| \neq |G|$  is of infinite global dim.

Pick  $\mathbb{Z}_2[C_2]$  as group algebra.  $\mathbb{Z}_2[C_2] \cong \mathbb{Z}_2[x]/\langle x^2 \rangle$ .

Image is local,  $[P] = 2[S] \rightarrow [P]$  don't generate  $K_0(A)$ .

Other sol.

$$K[\begin{matrix} 1 & \xrightarrow{a} & 2 \\ \xleftarrow{c} & & \end{matrix}] / \langle a^2, c^2 \rangle$$

$$P_1 = [K \begin{matrix} \xrightarrow{id} \\ \xleftarrow{id} \\ \circ \end{matrix} K], \quad P_2 = [K \begin{matrix} \xrightarrow{a} \\ \xleftarrow{id} \\ \circ \end{matrix} K]$$

These have the same dimension vector, so that is a problem.

Def  $(M, N) = \langle M, N \rangle + \langle N, M \rangle$ , this is a symmetric bilinear form.

$\mathcal{Q}$  acyclic

dimension vectors

Prop Let  $A = K\mathcal{Q}$ , then  $(x, y) = x^t C y$  is the Cartan matrix corresponding to the underlying graph of  $\mathcal{Q}$ .

Pr  $\text{Hom}(S_i, S_j) = \begin{cases} K & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$$\text{Ext}_A^1(S_i, S_j) = K^{\#\text{arrows from } j \text{ to } i}$$

$$S_i \leftarrow P_i \leftarrow \bigoplus_{\alpha: i \rightarrow j} P_{\text{start } \alpha}$$

$$\text{Hom}(S_i, S_j) : \text{Hom}(P_i, S_j) \xrightarrow{\sim} \text{Hom}(\bigoplus_{\alpha: i \rightarrow j} P_{\text{start } \alpha}, S_j) \rightarrow \text{Ext}_A^1(S_i, S_j)$$

$$\begin{cases} K & \text{if } \text{start } \alpha = j \\ 0 & \text{otherwise} \end{cases} \quad \square$$

Def.

The Cartan matrix of a finite dim. algebra  $A$  is

$$C_A = (\dim_{\mathbb{K}} \text{Hom}_A(P_i, P_j))_{i,j=1}^n$$

Note

$$C_A = (\dim P_1, \dots, \dim P_n) \quad \text{if } \text{End}(S_i) = \mathbb{K} \quad \forall i.$$

The  $i$ -th column of  $C_A$  is  $C_A \cdot e_i = (\dim_{\mathbb{K}} \text{Hom}(P_j, P_i))_{j=1}^n$  and  $\dim_{\mathbb{K}} \text{Hom}(P_j, M)$  is the multiplicity of  $S_j$  in the composition series of  $M$  (provided  $\text{End}(S_j) = \mathbb{K}$ ).

For injectives,  $C_A = (\dim_{\mathbb{K}} \text{Hom}_A(I_i, I_j))_{i,j=1}^n = \begin{pmatrix} \dim I_1^T \\ \vdots \\ \dim I_n^T \end{pmatrix}$

Obs

If gl. dim. of  $A$  is finite, then  $C_A$  is invertible. That is because  $\dim P_i$  is a basis for  $\mathbb{Z}^n$ .

Prop.

Assume finite gl. dim  $A$ . Then  $\langle x, y \rangle = x^T C^{-T} y$ .

Still assuming  $\text{End}(S_i) = \mathbb{K}$ .

Pf.

$$\langle \dim P_i, \dim S_j \rangle = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases}$$

$$\dim S_j = e_j, \quad \dim P_i = C_A e_i$$

$$\text{Let } M = (\langle e_i, e_j \rangle)_{i,j}^n. \quad (C_A e_i)^T M e_j = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases}$$
$$\parallel$$
$$e_i^T C_A^T M e_j \Rightarrow C_A^T M = I.$$

$$\text{so } M = C_A^{-T} \quad \square$$

Cor

$Q$  is an acyclic quiver. The Cartan matrix corresponding to the underlying graph of  $Q$  is  $C_{\mathbb{K}Q}^{-1} + C_{\mathbb{K}Q}^{-T}$ .

Def.

$x \in \mathbb{Z}^n$  is called a root for  $\langle -, - \rangle$  if  $\langle x, x \rangle = 1$ .

Ex

Every  $e_i$  is a root for  $\langle -, - \rangle_{\mathbb{K}Q}$

Obs

If  $\langle -, - \rangle$  is positive definite, then there are only finitely many roots.

Let  $\langle x, x \rangle$  be a root,  $\langle x, x \rangle = |x|^2 \langle \frac{x}{|x|}, \frac{x}{|x|} \rangle$ ,  $x \neq 0$

Look at  $\{\langle x, x \rangle \mid |x|=1 \text{ and } x \in \mathbb{R}^n\}$ , this is a compact image of the unit sphere in  $\mathbb{R}^n$ . So there is a min.  $m > 0$

$\langle x, x \rangle = |x|^2 \langle \frac{x}{|x|}, \frac{x}{|x|} \rangle \geq |x|^2 m$ . Since  $x$  is a root, then  $|x| \leq \frac{1}{\sqrt{m}}$ . There are only finitely many integers less than  $\frac{1}{\sqrt{m}}$ .

End of lecture ~ Start of lecture

Can we describe the dim. vec. of  $\tau M$  in terms of the dim. vec. of  $M$ ?

Note

Let  $\rho \in \text{proj } KQ$ , then  $\frac{\dim v\rho}{\dim \rho} = C_A^{-1} C_A \dim \rho$

$$\begin{aligned} \dim \rho_i &= C_A \dim S_i \\ \dim I_i &= C_A^T \dim S_i \\ &= C_A^T C_A^{-1} \dim \rho_i \end{aligned}$$

Calculate  $\tau$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & Q & \rightarrow & P & \rightarrow & M \rightarrow 0 \\ & & & & \downarrow & & \\ 0 & \rightarrow & \tau M & \rightarrow & vQ & \rightarrow & vP \rightarrow vM \rightarrow 0 \end{array}$$

$\circ$  unless  $M$  contains a proj. summand.

$$\dim \tau M = \dim vQ - \dim vP = C_A^T C_A^{-1} (\dim Q - \dim P) = C_A^T C_A^{-1} \dim \rho$$

Def. The Coxeter - transformation

$$\Phi = -C_A^T C_A^{-1}$$

Prop

$M$  is an indec. module over  $KQ$ , then

$$\Phi(\dim M) = \begin{cases} \dim \tau M & \text{if } M \text{ not proj,} \\ -\dim \tau M & \text{otherwise.} \end{cases}$$

Thm

Indec. preprojective and preinjective  $\mathbb{K}Q$ -modules are uniquely determined by the dimension vectors.

Pf

$M$  is preprojective  $\Leftrightarrow \exists n$  s.t.  $\tau^n M$  is projective.  
 $\Leftrightarrow \exists n \Phi^{n+1}(\underline{\dim} M) \geq 0$  a non-positive vector.

If  $\exists n \geq 0 \Phi^{n+1}(\underline{\dim} M)$  is non-positive, pick a minimal s.t. this happens.  $\rightarrow \Phi^n(\underline{\dim} M) = \underline{\dim} P_i$  for some  $i$ .  $\underline{\dim} P_i$  uniquely determines  $P_i$ , so  $M = \tau^{-n} P_i$ .  $\square$

Thm

If  $M$  is pre-projective or -injective over a quiver algebra  $\mathbb{K}Q$ , then  $\langle \underline{\dim} M, \underline{\dim} M \rangle = 1$ , i.e.  $\underline{\dim} M$  is a root.

Pf

If  $M$  is indec. proj., then  $\dim(\text{Hom}(M, M)) = 1$ ,  $\dim(\text{Ext}^1(M, M)) = 0$ .  
 $\Rightarrow \langle \underline{\dim} M, \underline{\dim} M \rangle = 1$ .

$$\begin{aligned} \langle \underline{\dim} \tau M, \underline{\dim} \tau M \rangle &= (\Phi \underline{\dim} M)^T C_A^{-T} (\Phi \underline{\dim} M) \\ &= (-C_A^T C_A^{-1} \underline{\dim} M)^T C_A^{-T} (-C_A^{-T} C_A \underline{\dim} M) \\ &= \underline{\dim} M^T C_A^T C_A C_A^T C_A C_A^{-1} \underline{\dim} M \\ &= \underline{\dim} M^T C_A^T \underline{\dim} M \\ &= \langle \underline{\dim} M, \underline{\dim} M \rangle. \end{aligned} \quad \square$$

Cor

$\mathbb{K}Q$ ,  $Q$  is Dynkin, then  $\mathbb{K}Q$  is representation finite.

Pf

$Q$  Dynkin  $\Leftrightarrow C_Q$  is positive definite  $\Leftrightarrow \langle -, - \rangle$  is positive def.  
 $\Leftrightarrow \langle -, - \rangle$  is positive definite  
 $\Rightarrow$  There are only finitely many roots of  $\langle -, - \rangle$   
 $\Rightarrow$  There are only finitely many preprojectives  
 $\Rightarrow \mathbb{K}Q$  is rep. finite.

Thm

If  $Q$  is Dynkin, then  $\{\text{indec } \mathbb{K}Q\text{-modules}\} / \cong \xrightarrow{\underline{\dim}} \{\text{non-neg. roots of } \langle -, - \rangle\}$  is a bijection.

Pf

We have seen well-defined and injective. Let  $x$  be a non-negative root. Note  $\exists M$  module s.t.  $\underline{\dim} M = x$ , e.g. some s.s.  $M$ .

Pick  $M$  s.t.  $\underline{\dim} M = x$  with smallest possible  $\text{End}(M)$ .

Assuming  $M \cong M_1 \oplus M_2$  s.t.  $M_i \neq 0$ , then  $\langle \dim M, \dim M \rangle = 1$   
 $= \langle \dim M_1, \dim M_1 \rangle + \langle \dim M_1, \dim M_2 \rangle + \langle \dim M_2, \dim M_1 \rangle$   
 $+ \langle \dim M_2, \dim M_2 \rangle$

Know that  $\langle \dim M_i, \dim M_i \rangle > 0$ , so assume that  
 $\langle \dim M_1, \dim M_2 \rangle < 0 \Rightarrow \text{Ext}^1(M_1, M_2) \neq 0$ .

Consider  $0 \rightarrow M_2 \rightarrow E \rightarrow M_1 \rightarrow 0$  non-split s.e.s.  
 We hom the seq. to itself.

$$\begin{array}{ccccccc} \text{Hom}(M_1, M_2) & \twoheadrightarrow & \text{Hom}(M_1, E) & \rightarrow & \text{Hom}(M_1, M_1) & \xrightarrow{\cong} & \text{Ext}^1(M_1, M_2) \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Hom}(E, M_2) & \twoheadrightarrow & \text{Hom}(E, E) & \rightarrow & \text{Hom}(E, M_1) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Hom}(M_2, M_2) & \twoheadrightarrow & \text{Hom}(M_2, E) & \rightarrow & \text{Hom}(M_2, M_1) & & \end{array}$$

$$\begin{aligned} \dim \text{Hom}(E, E) &\leq \underbrace{\dim(\text{Hom}(M_1, E))}_{\leq \dim \text{Hom}(M_1, M_2)} + \underbrace{\dim(\text{Hom}(M_2, E))}_{\leq \dim \text{Hom}(M_2, M_1)} \\ &\leq \dim \text{Hom}(M_1, M_2) + \dim \text{Hom}(M_2, M_1) \\ &\leq \dim \text{Hom}(M_1, M_1) + \dim \text{Hom}(M_2, M_2) \\ &\leq \dim \text{Hom}(M, M) \quad \downarrow \text{choice of } M \\ &\Rightarrow M \text{ is indec.} \quad \square \end{aligned}$$

Rem

$Q$  is Dynkin, and  $\langle x, x \rangle = 1$ , the  $x$  is non-positive or non-negative.

Def

An indecomposable  $KQ$ -module that is neither pre-projective nor -injective is called regular.

Fact

If  $Q$  is non-Dynkin, then there are regular modules.

Sketch

Pick  $x$  as a positive vector s.t.  $\langle x, x \rangle \leq 0$  and minimal with this property.

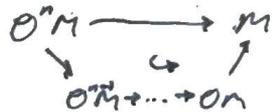
Copy the same argument for last proof in order to find indec  $M$  s.t.  $\dim M = x$ .

if  $M \cong M_1 \oplus M_2$ , then  
 $\dim M_1, \dim M_2 \leq \dim M$   
 $\Rightarrow \langle \dim M_1, \dim M_1 \rangle > 0$   
 Assumption ok

Describing  $\text{rad}^n \text{mod}^A$

For  $M$  an indec. module, we describe by  $\theta M \rightarrow M$  the right minimal almost split map ending in  $M$ . Extend this by setting  $\theta(\theta M_i) = \theta \theta M_i$ .

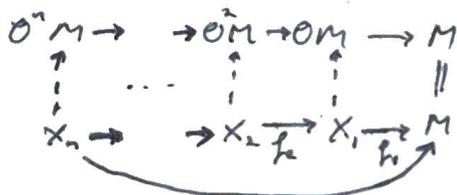
Lemma  $f: N \rightarrow M$ , then  $f \in \text{rad}^n(N, M)$  iff  $f$  factors through



Pf  $\theta^n M \rightarrow M \in \text{rad}^n(\theta^n M, M)$

" $\Leftarrow$ "

Let  $f \in \text{rad}^n(N, M)$ , then  $f = f_n \circ \dots \circ f_1$  for  $f_i \in \text{rad}(N, M)$



Lemma  $M \in \text{mod}^A$ , if  $\text{rad}^{n+1}(-, M) = \text{rad}^n(-, M)$ , then both are 0.

Rem This is not true if  $M$  was in the contravariant argument

Pf  $\theta^{n+1} M \xrightarrow{b} \theta^n M \xrightarrow{a} M \quad a \in \text{rad}^n$

If  $a$  is also in  $\text{rad}^{n+1}$ , then  $a$  factors through  $ab$ , i.e.  $a = ab \circ c \Rightarrow a = 0$

$\in \text{rad}$   
 $\Rightarrow$  nilpotent

$$\begin{aligned} \text{rad}^n(-, M) &= \{ f \mid f \text{ factors through } a \} \\ &= \{ f \mid f \text{ factors through } 0 \} = 0. \end{aligned}$$

□