Repteori - Week 10

Exercises

1. $\mathbb{K} \subseteq \mathbb{L}$ where $[\mathbb{L} : \mathbb{K}] = 3$ are fields. Find the AR-quiver of the algebra:

$$\begin{pmatrix} \mathbb{K} & \mathbb{L} \\ 0 & \mathbb{L} \end{pmatrix} = A.$$

We know that $A \simeq \underset{P_1}{(\mathbb{K}\ \mathbb{L})} \oplus \underset{P_2}{(0\ \mathbb{L})}$.
Moreover $\operatorname{rad} P_1 = P_2$.

$$\begin{tikzcd}
 & P_1 \arrow[dr, "{(3,1)}"] & & (\mathbb{K}^2\ \mathbb{L}) \arrow[ll, dashed] \arrow[dr, "{(1,3)}"] & & i_1 \arrow[ll, dashed] \\
P_2 \arrow[ur, "{(1,3)}"] & & (\mathbb{K}^3\ \mathbb{L}^2) \arrow[ll, dashed] \arrow[ur, "{(3,1)}"] & & i_2 \arrow[ll, dashed] \arrow[ur, "{(3,1)}"'] &
\end{tikzcd}$$

The injectives are $i_1 = (\mathbb{K}\ \ 0)$ and $i_2 = (\mathbb{L}\ \ \mathbb{L})$.

$\tau^- P_2 : \quad P_2 \rightarrowtail i_2 \longrightarrow i_1^3$

Calculating τ^-

$$\Big\downarrow$$

$$P_2 \rightarrow P_1^3 \longrightarrow (\mathbb{K}^3\ \mathbb{L}^2)$$

2. Assume there is an arrow $M \xrightarrow{(a,b)} N$ in the AR-quiver, s.t. $a, b \geq 2$. To this end we may assume $a, b = 2$.

$$\begin{tikzcd}
 & & R \arrow[dr] & \\
 & M \arrow[ur] \arrow[dr, "{(a,b)}"] & & \tau^- M \arrow[ll, dashed] \\
\tau N \arrow[ur, "{(b,a)}"] \arrow[dr] & & N \arrow[ll, dashed] \arrow[ur, "{(b,a)}"'] & \\
 & R' \arrow[ur] & &
\end{tikzcd}$$

$$\begin{aligned}
\dim M + \dim \tau^- M &= b \dim N + \dim R \\
&\geq 2 \dim N \\
\dim N + \dim \tau N &= a \dim M + \dim R' \\
&\geq 2 \dim M
\end{aligned}$$

$\dim M \neq \dim N$ irr. mor. is either mono or epi.

Assume $\dim M < \dim N \Rightarrow \dim \tau^- M > \dim N$.
Use this inductively to get that $\dim (\tau^-)^n M > \dim (\tau^-)^{(n-1)} M$ for any n. M is not injective

Analogously if $\dim N < \dim M$.

dimension vector of the projectives

3. Find an algebra A s.t. $([P_i])_{i=1}^{n}$ is a basis for $K_0(\mathrm{mod}^A)$.

We know that $\mathrm{gl.dim}\, A = \infty$, otherwise we find every module as an alternating sum of it's projective resolution.

Fact
Every group algebra s.t. $|\mathbb{K}| \mid |G|$ is of infinite global dim.

Pick $\mathbb{Z}_2[C_2]$ as group algebra. $\mathbb{Z}_2[C_2] \simeq \mathbb{Z}_2[x]/\langle x^2 \rangle$.

Image is local, $[P] = 2[S] \to [P]$ don't generate $K_0(A)$.

Other sol.

$$\mathbb{K}\left[\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}\right] / \langle ab, ba \rangle$$

$$P_1 = \left[\begin{tikzcd} \mathbb{K} \arrow[r, bend left, "id"] & \mathbb{K} \arrow[l, bend left, "0"] \end{tikzcd}\right], \quad P_2 = \left[\begin{tikzcd} \mathbb{K} \arrow[r, bend left, "0"] & \mathbb{K} \arrow[l, bend left, "id"] \end{tikzcd}\right]$$

These have the same dimension vector, so that is a problem.

Def $(M, N) = \langle M, N \rangle + \langle N, M \rangle$, this is a symmetric bilinear form.

Q acyclic dimension vectors

Prop Let $A = \mathbb{K}Q$, then $(x, y) = x^t C y$ is the Cartan matrix corresponding to the underlying graph of Q.

Pf.
$$\mathrm{Hom}(S_i, S_j) = \begin{cases} \mathbb{K} & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^1_A(S_i, S_j) = \mathbb{K}^{\#\text{arrows from } j \text{ to } i}$$

$$S_i \twoheadleftarrow P_i \leftarrow \bigoplus_{\alpha : ? \to i} P_{\mathrm{start}(\alpha)}$$

$$\mathrm{Hom}(-, S_j) : \quad \mathrm{Hom}(P_i, S_j) \xrightarrow{0} \underbrace{\mathrm{Hom}\Big(\bigoplus_{\alpha : ? \to i} P_{\mathrm{start}(\alpha)}, S_j\Big)}_{\substack{\mathbb{K} \text{ if } \mathrm{start}(\alpha) = j \\ 0, \text{ otherwise}}} \twoheadrightarrow \mathrm{Ext}^1_A(S_i, S_j)$$

$\square$

Def.
The Cartan matrix of a finite dim. algebra A is

$$C_A = (\dim_{\mathbb{K}} \operatorname{Hom}_A(P_i, P_j))_{i,j=1}^n$$

Note
$$C_A = (\underline{\dim}\, P_1, \dots, \underline{\dim}\, P_n), \quad \text{if } \operatorname{End}(S_i) = \mathbb{K} \ \ \forall i.$$

The i-th column of C_A is $C_A \cdot e_i = (\dim_{\mathbb{K}} \operatorname{Hom}(P_j, P_i))_{j=1}^n$ and $\dim_{\mathbb{K}} \operatorname{Hom}(P_j, M)$ is the multiplicity of S_j in the composition series of M (provided $\operatorname{End}(S_j) = \mathbb{K}$).

For injectives, $C_A = (\dim_{\mathbb{K}} \operatorname{Hom}_A(I_i, I_j))_{i,j=1}^n = \begin{pmatrix} \underline{\dim}\, I_1^T \\ \vdots \\ \underline{\dim}\, I_n^T \end{pmatrix}$

Obs
If gl. dim. of A is finite, then C_A is invertible. That is because $\underline{\dim}\, P_i$ is a basis for $\mathbb{Z}^n$.

Prop.
Assume finite gl. dim A. Then $\langle x, y \rangle = x^T C^{-T} y$.

Still assuming $\operatorname{End}(S_i) = \mathbb{K}$.

Pf.
$$\langle \underline{\dim}\, P_i, \underline{\dim}\, S_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

$$\underline{\dim}\, S_j = e_j, \quad \underline{\dim}\, P_i = C_A e_i$$

Let $M = (\langle e_i, e_j \rangle)_{i,j}^n$. $(C_A e_i)^T M e_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$

$$(C_A e_i)^T M e_j = e_i^T C_A^T M e_j \Rightarrow C_A^T M = I.$$

so $M = C_A^{-T}$ □

Cor
Q is an acyclic quiver. The Cartan matrix corresponding to the underlying graph of Q is $C_{\mathbb{K}Q}^{-1} + C_{\mathbb{K}Q}^{-T}$.

Def.
$x \in \mathbb{Z}^n$ is called a root for $\langle -, - \rangle$ if $\langle x, x \rangle = 1$.

Ex
Every e_i is a root for $\langle -, - \rangle_{\mathbb{K}Q}$

Obs

If $\langle -,- \rangle$ is positive definite, then there are only finitely many roots.

Let $\langle x,x \rangle$ be a root, $\langle x,x \rangle = |x|^2 \langle \overset{\text{length } 1}{\frac{x}{|x|}, \frac{x}{|x|}} \rangle$, $x \neq 0$

Look at $\{\langle x,x \rangle \mid |x| = 1 \text{ and } x \in \mathbb{R}^n\}$, this is a compact image of the unit sphere in $\mathbb{R}$. So there is a min. $m > 0$

$\langle x,x \rangle = |x|^2 \langle \frac{x}{|x|}, \frac{x}{|x|} \rangle \geq |x|^2 m$. Since x is a root, then $|x| \leq 1/\sqrt{m}$. There are only finitely many integers less than $1/\sqrt{m}$.

End of lecture ~ Start of lecture

Can we describe the dim. vec. of τM in terms of the dim. vec. of M?

Note

Let $P \in \operatorname{proj} \mathbb{K}Q$, then $\underline{\dim}\, \nu P = C_A^T C_A \underline{\dim}\, P$.

$\underline{\dim}\, P_i = C_A \underline{\dim}\, S_i$

$\underline{\dim}\, I_i = C_A^T \underline{\dim}\, S_i = C_A^T C_A^{-1} \underline{\dim}\, P_i$

Calculate τ:

$$0 \to Q \to P \to M \to 0$$

$$\Big\updownarrow \nu$$

$$0 \to \tau M \to \nu Q \to \nu P \to \underbrace{\nu M}_{\substack{0 \text{ unless } M \\ \text{contains a proj.} \\ \text{summand.}}} \to 0$$

$$\underline{\dim}\, \tau M = \underline{\dim}\, \nu Q - \underline{\dim}\, \nu P = C_A^T C_A^{-1} (\underline{\dim}\, Q - \dim P) = C_A^T C_A^{-1} \underline{\dim}\, P$$

Def. The Coxeter-transformation

$$\Phi = -C_A^T C_A^{-1}$$

Prop M is an indec. module over $\mathbb{K}Q$, then

$\Phi(\underline{\dim}\, M) = \{ \underline{\dim}\, \tau M$ if M not proj, $-\dim \tau M$ otherwise.

Thm
Indec. preprojective and pre injective $\underbrace{}_{}$ $\mathbb{K}Q$-modules are uniquely determined by the dimension vectors.

Pf M is preprojective $\Leftrightarrow \exists n$ s.t. $\tau^n M$ is projective.
$\Leftrightarrow \exists n\ \Phi^{n+1}(\underline{\dim}\, M)$ is a non-positive vector.

If $\exists n \geq 0\ \Phi^{n+1}(\underline{\dim}\, M)$ is non-positive, pick n minimal s.t. this happens. $\rightsquigarrow \Phi^n(\underline{\dim}\, M) = \underline{\dim}\, P_i$ for some i. $\underline{\dim}\, P_i$ uniquely determines P_i, so $M = \tau^{-n} P_i$ $\square$

Thm
If M is pre-projective or -injective over a quiver algebra $\mathbb{K}Q$, then $\langle \underline{\dim}\, M, \underline{\dim}\, M \rangle = 1$, i.e. $\underline{\dim}\, M$ is a root.

Pf If M is indec. proj., then $\dim(\mathrm{Hom}(M,M)) = 1$, $\dim(\mathrm{Ext}^1(M,M)) = 0$.
$\Rightarrow \langle \underline{\dim}\, M, \underline{\dim}\, M \rangle = 1$.

$$\begin{aligned}
\langle \underline{\dim}\, \tau M, \underline{\dim}\, \tau M \rangle &= (\Phi\, \underline{\dim}\, M)^T C_A^{-T} (\Phi\, \underline{\dim}\, M) \\
&= (-C_A^T C_A^{-1} \underline{\dim}\, M)^T C_A^{-T} (-C_A^T C_A^{-1} \underline{\dim}\, M) \\
&= \underline{\dim}\, M^T C_A^{-T} C_A C_A^{-T} C_A^T C_A^{-1} \underline{\dim}\, M \\
&= \underline{\dim}\, M^T C_A^{-T} \underline{\dim}\, M \\
&= \langle \underline{\dim}\, M, \underline{\dim}\, M \rangle. \qquad \square
\end{aligned}$$

Cor
$\mathbb{K}Q$, Q is Dynkin, then $\mathbb{K}Q$ is representation finite.

Pf Q Dynkin $\Leftrightarrow C_Q$ is positive definite $\Leftrightarrow (-,-)$ is positive def.
$\Leftrightarrow \langle -,- \rangle$ is positive definite
$\Rightarrow$ There are only finitely many roots of $\langle -,- \rangle$
$\Rightarrow$ There are only finitely many preprojectives
$\Rightarrow \mathbb{K}Q$ is rep. finite.

Thm
If Q is Dynkin, then $\{\text{indec } \mathbb{K}Q\text{-modules}\}/\cong \xrightarrow{\underline{\dim}} \{\text{non-neg. roots of } \langle -,- \rangle\}$ is a bijection.

Pf We have seen well-defined and injective. Let x be a non-negative root. Note $\exists M$ module s.t. $\underline{\dim}\, M = x$, e.g. some s.s. M.

Pick M s.t. $\underline{\dim}\, M = x$ with smallest possible $\mathrm{End}(M)$.

Assuming $M \simeq M_1 \oplus M_2$ s.t. $M_i \neq 0$, then $\langle \underline{\dim}\, M, \underline{\dim}\, M \rangle = 1$
$= \langle \underline{\dim}\, M_1, \underline{\dim}\, M_1 \rangle + \langle \underline{\dim}\, M_1, \underline{\dim}\, M_2 \rangle + \langle \underline{\dim}\, M_2, \underline{\dim}\, M_1 \rangle + \langle \underline{\dim}\, M_2, \underline{\dim}\, M_2 \rangle$

Know that $\langle \underline{\dim}\, M_i, \underline{\dim}\, M_i \rangle > 0$, so assume that $\langle \dim M_1, \dim M_2 \rangle < 0 \Rightarrow \operatorname{Ext}^1(M_1, M_2) \neq 0$.

Consider $0 \to M_2 \to E \to M_1 \to 0$ non-split s.e.s.
We hom the seq. to itself.

$$\begin{tikzcd}
\operatorname{Hom}(M_1, M_2) \arrow[r, tail] \arrow[d, tail] & \operatorname{Hom}(M_1, E) \arrow[r] \arrow[d, tail] & \operatorname{Hom}(M_1, M_1) \arrow[r, "\neq 0"] \arrow[d, tail] & \operatorname{Ext}^1(M_1, M_2) \\
\operatorname{Hom}(E, M_2) \arrow[r, tail] \arrow[d] & \operatorname{Hom}(E, E) \arrow[r] \arrow[d] & \operatorname{Hom}(E, M_1) \arrow[d] & \\
\operatorname{Hom}(M_2, M_2) \arrow[r, tail] & \operatorname{Hom}(M_2, E) \arrow[r] & \operatorname{Hom}(M_2, M_1) &
\end{tikzcd}$$

$$\dim \operatorname{Hom}(E,E) \leq \underbrace{\dim(\operatorname{Hom}(M_1, E))}_{\substack{\leq \dim \operatorname{Hom}(M_1, M_2) \\ + \dim(\operatorname{Hom}(M_1, M_1))}} + \underbrace{\dim(\operatorname{Hom}(M_2, E))}_{\substack{\leq \dim \operatorname{Hom}(M_2, M_1) \\ + \dim \operatorname{Hom}(M_2, M_2)}}$$

$< \dim \operatorname{Hom}(M, M)$ ↯ choice of M
$\Rightarrow M$ is indec. □

Rem
Q is Dynkin, and $\langle x, x \rangle = 1$, the x is non-positive or non-negative.

Def
An indecomposable $\mathbb{K}Q$-module that is neither pre-projective nor -injective is called regular.

Fact
If Q is non-Dynkin, then there are regular modules.

Sketch
Pick x as a positive vector s.t. $\langle x, x \rangle \leq 0$ and minimal with this property.

Copy the same argument for last proof in order to find indec M s.t. $\underline{\dim}\, M = x$.

if $M \cong M_1 \oplus M_2$, then
$\underline{\dim}\, M_1, \underline{\dim}\, M_2 < \underline{\dim}\, M$
$\Rightarrow \langle \underline{\dim}\, M_i, \underline{\dim}\, M_i \rangle > 0$.
Assumption

ok

Describing $\mathrm{rad}^n \mathrm{mod}^A$

For M an indec. module, we describe by $\Theta M \to M$ the right minimal almost split map ending in M. Extend this by setting $\Theta(\oplus M_i) \simeq \oplus \Theta M_i$.

Lemma
$f: N \to M$, then $f \in \mathrm{rad}^n(N, M)$ iff f factors through

$$\begin{tikzcd}
\Theta^n M \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & M \\
& \Theta^{n-1}M \to \dots \to \Theta M \arrow[ur] &
\end{tikzcd}$$

Pf
$\Theta^n M \to M \in \mathrm{rad}^n(\Theta^n M, M)$

"$\Leftarrow$"

Let $f \in \mathrm{rad}^n(N, M)$, then $f = f_1 \circ \dots \circ f_n$ for $f_i \in \mathrm{rad}(N, M)$

$$\begin{tikzcd}
\Theta^n M \arrow[r] & \cdots \arrow[r] & \Theta^2 M \arrow[r] & \Theta M \arrow[r] & M \arrow[d, equal] \\
X_n \arrow[u, dashed] \arrow[r] \arrow[rrrr, bend right] & \cdots \arrow[r] & X_2 \arrow[u, dashed] \arrow[r, "f_2"'] & X_1 \arrow[u, dashed] \arrow[r, "f_1"'] & M
\end{tikzcd}$$

Lemma
$M \in \mathrm{mod}^A$, if $\mathrm{rad}^{n+1}(-, M) = \mathrm{rad}^n(-, M)$, then both are 0.

Rem
This is not true if M was in the contravariant argument

Pf

$$\begin{tikzcd}
\Theta^{n+1}M \arrow[r, "b"] & \Theta^n M \arrow[r, "a"] \arrow[l, bend left, "c"] & M
\end{tikzcd} \qquad a \in \mathrm{rad}^n$$

If a is also in $a \in \mathrm{rad}^{n+1}$, then a factors through ab, i.e. $a = a\underbrace{bc}_{\substack{\in\, \mathrm{rad} \\ \Rightarrow \text{ nilpotent}}} \Rightarrow a = 0$

$\mathrm{rad}^n(-, M) = \{ f \mid f \text{ factors through } a \}$
$= \{ f \mid f \text{ factors through } 0 \} = 0.$ $\square$