Repteori - Week 11

Exercises

2. Find the matrix Φ for:

What are the eigenvalues / Jordan form?

a) $\mathbb{K}[1 \to 2]$

$$\Phi = -C_A^T C_A^{-1} \qquad C_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \Phi = -\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

eigenvalues are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

b) $\mathbb{K}[1 \rightrightarrows 2]$, $C_A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$\Phi = -\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

eigenvalues are 1, so Jordan form is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

note: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

1. Find a regular decomposition for:

a) $\mathbb{K}[1 \to 2]$

A module with dimension vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is regular, as $\Phi^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \not\geq 0$ for any n, nor $\Phi^{-n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \not\geq 0$.

b) $\mathbb{K}\left\{ \begin{tikzcd} & & 0 \arrow[dll, "a"'] \arrow[dl, "b"] \arrow[d, "c"] \arrow[dr, "d"] & \\ 1 & 2 & 3 & 4 \end{tikzcd} \right\}$ Pick $\begin{tikzcd} & & \mathbb{K}^2 & \\ \mathbb{K} \arrow[urr, "\binom{1}{0}"] & \mathbb{K} \arrow[ur, "\binom{0}{1}"] & \mathbb{K} \arrow[u, "\binom{1}{1}"] & \mathbb{K} \arrow[ul, "\binom{1}{\lambda}"'] \end{tikzcd}$

Let $f = \begin{array}{cccc} & 2 & & \\ 1 & 1 & 1 & 1 \end{array}$ be an additive function, then

$$c_Q f = 0, \quad (-, f) = 0, \quad \langle f, f \rangle = 0$$

3.

a) If Q is Dynkin, then $\Phi^n = 1$ for some n.
Let x be a root, then there is some n s.t. $\Phi^n x = x \leftarrow \Phi x$ is also a root, but there are only finitely many roots. So $\Phi|_{\text{roots}}$ have finite order, and since the roots generate the Grothendieck group, Φ is of finite order.

b) Let Q be Euclidean and f an additive function, show that $\forall x \in \mathbb{Z}^n \; \exists n > 0, m \in \mathbb{Z}$ s.t. $\Phi^n x = x + mf$.

f is additive $\Leftrightarrow (-, f) = 0$
$\rightsquigarrow$ induces a symmetric bilinear form on $\mathbb{Z}^r/(f)$, where $r = \#$vertices

$\mathbb{Z}^r/(f) \simeq \mathbb{Z}^{r-1}$ This is the Dynkin diagram inside the Euclidean diagram.

We get a bil. form $(-,-)_{K.Dyn.}$ which is positive definite. We can subtract the additive function, s.t. the circled vertex is 0.

Φ is well-defined on $\mathbb{Z}^r/(f)$; $\Phi f = -C_{KQ}^{-T} C_{KQ}^{-1} f = f$
as $(-, f) = (C_{KQ}^{-T} + C_{KQ}^{-1}) f = 0$.

By a) Φ has finite order in $\mathbb{Z}^r/(f)$.

Lemma
If M, N are indec. and there is an arrow $M \to N$ in the AR-quiver, then $\dim M \leq (1 + (\dim A)^2) \dim N$

Pf.

$$\tau N: \quad \begin{tikzcd} P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[rr, two heads] & & N \\ & \Omega N \arrow[ur] \arrow[d, "\nu"] & & & \\ \tau M \arrow[r, tail] & \nu P_1 \arrow[r] & \nu P_0 & & \end{tikzcd}$$

Number of summands of in $P_0 \leq \dim N$
$\rightsquigarrow \dim P_0 \leq \dim N \cdot \dim A$
$\rightsquigarrow \dim \Omega N \leq \dim N \cdot \dim A$
$\rightsquigarrow \#$ summands in $P_1 \leq \dim N \cdot \dim A$
$\#$ summands of $\nu P_1 \leq \dim N \cdot \dim A$
$\rightsquigarrow \dim \nu P_1 \leq \dim N \cdot (\dim A)^2$
$\rightsquigarrow \dim \tau N \leq \dim N \cdot (\dim A)^2$

$$\begin{aligned} \dim M &\leq \dim N + \dim \tau N \\ &\leq \dim N (1 + (\dim A)^2) \end{aligned}$$

$\square$

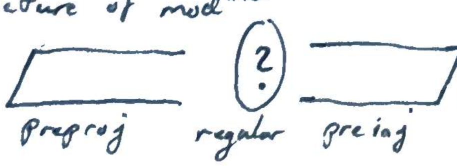
Regular modules over hereditary algebras

Obs $\mathrm{Hom}_{KQ}(M, N) = 0$ if either:

- M regular, N preproj
- M preinj, N preproj
- M preinj, N regular

N preproj, M not preproj. Any map from M to N factors through a right minimal almost split map, $\Theta N \to N$, and then through $\Theta^2 N$, $\Theta^3 N$ and so on. Since N is preproj, $\Theta^n N = 0$ eventually.

Picture of mod^{KQ}



maps only goes from left to right

Lemma

M is regular $\rightsquigarrow$ $\nexists$ proper (not iso) mono or epi $\tau M \to M$

Pf.

Assume a proper mono exists

$$0 \to \tau M \to M \to \mathrm{Cok} \to 0$$

Claim: $\tau = D\mathrm{Ext}^1(-, \mathbb{K}Q)$

$$M \twoheadleftarrow P_0 \leftarrow P_1$$

$$\Big\downarrow \nu = D\mathrm{Hom}(-, \mathbb{K}Q)$$

$$\nu M \twoheadleftarrow \nu P_0 \leftarrow \nu P_1 \leftarrow D\mathrm{Ext}^1(M, \mathbb{K}Q) \leftarrow \overset{0}{\overset{\|}{D\mathrm{Ext}^1(P_0, \mathbb{K}Q)}}$$

$$\Rightarrow D\mathrm{Ext}^1(M, \mathbb{K}Q) \simeq \tau M$$

$\mathrm{Hom}(-, \mathbb{K}Q)$ vanishes on the sequence.
τ is exact on the sequence. so

$$0 \to \tau^2 M \to \tau M \to \tau\,\mathrm{Cok} \to 0$$

We iterate this as, with every morphism as a proper mono.

$$\cdots \hookrightarrow \tau^n M \hookrightarrow \cdots \hookrightarrow \tau^2 M \hookrightarrow \tau M \hookrightarrow M \quad ↯ \; M \text{ is f.d.} \; \square$$

Lemma

M is regular, let there be an almost split seq.

$$0 \to \tau M \xrightarrow{\binom{f_1}{f_2}} E \oplus F \xrightarrow{(g_1\ g_2)} M \to 0$$

Then:

- f_1 mono, g_1 epi, f_2 epi and g_2 mono
- f_1 epi, g_1 mono, f_2 mono, and g_2 epi

Pf.

f_1 mono $\Leftrightarrow$ g_1 epi — Previous lemma

g_2 mono $\Leftrightarrow$ f_2 epi

$$\begin{tikzcd}
\tau M \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\ulcorner" very near start, "\lrcorner" very near end] & E \arrow[d, "g_1"] \\
F \arrow[r, "g_2"'] & M
\end{tikzcd}$$

g_1 mono/epi $\Leftrightarrow$ f_2 mono/epi

f_1 mono/epi $\Leftrightarrow$ g_2 mono/epi

Lemma

M regular, and an almost split seq.

$$\tau M \xrightarrow{\binom{f_1}{f_2}} E \oplus F \xrightarrow{(g_1\ g_2)} M$$

If f_2 is epi, then F is either indec. or 0

Pf.

Assume $F \simeq \bigoplus_{i=1}^{r} F_i$ w/ F_i indec.

$\dim F < \dim M$, $\dim F < \dim \tau M$

$\tau M \twoheadrightarrow F \rightsquigarrow M \twoheadrightarrow \tau^{-} F \rightsquigarrow \dim \tau^{-} F < \dim M$

For each F_i, M appears a summand in the almost split sequence starting in F_i.

$r \dim M \leq \dim F + \dim \tau^{-} F < \dim M + \dim M$

$\Rightarrow r < 2$ $\qquad$ $r = 1$ is the only choice. □

End of lecture ~

Lemma

Let M be regular

$$0 \to \tau M \xrightarrow{(f_i)} \bigoplus_{i=1}^{r} E_i \xrightarrow{(g_i)} M \to 0 \quad \text{almost split, } E_i \text{ indec.}$$

Then $r \leq 3$

If $r = 3$, then all f_i are epi and all g_i are mono

Pf.

Assume $r \geq 3$

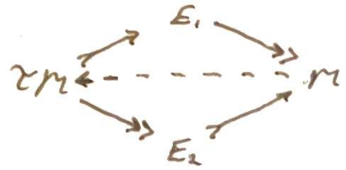
$$0 \to \tau M \xrightarrow{\begin{pmatrix} \binom{f_1}{f_2} \\ (f_i) \end{pmatrix}} \underbrace{(E_1 \oplus E_2)}_{E} \oplus \underbrace{\left(\bigoplus_{i \geq 3} E_i\right)}_{F} \xrightarrow{((g_1\ g_2)(g_i))} M$$

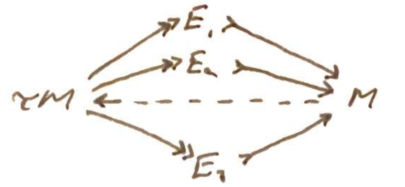
E is not indec. nor 0, so $\binom{f_1}{f_2}$ is mono and (f_i) is epi, so F is indec. Thus $r = 3$ and f_3 is epi and g_3 is mono.

By symmetry, all f_i are epi and all g_i are mono □

There are 3 possibilities for "all" arrows in the AR-quiver among the regular modules.

1. 

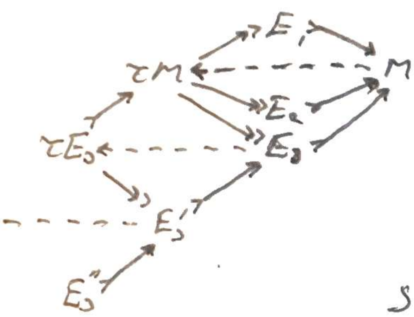
2. 

3. 

This is the complete list if $\mathbb{K}$ is alg. closed. Some E_i, E_j may be isomorphic if we not assume this.

Two possible AR-component of regular modules:

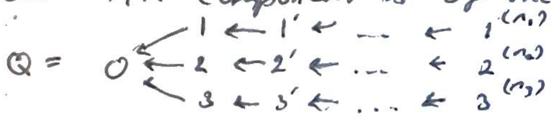
If an almost split sequence as in 3. appears



E_i cannot have type 3. AR-seq.
If E_3 has type 2. we get the net as indicated

We get $M \leftarrow E_3 \leftarrow E_3' \leftarrow E_3'' \leftarrow \dots$
Since M is f.d. $E_3^{(n)} = 0$ somewhere.

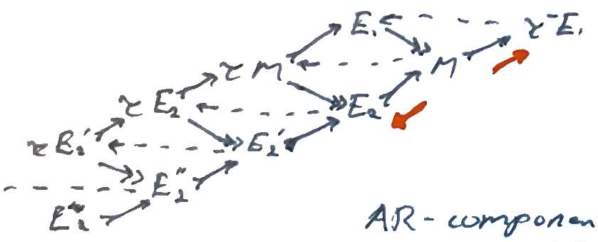
So it switches to type 1. at some point.
Our AR-component is of the form $\mathbb{Z}Q/G$ where



Remark

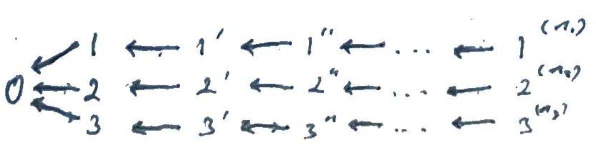
$G = 0$, otherwise the component is finite
If our algebra is connected, then if there is a finite component, it is the whole AR-quiver

If no almost split seq as in 3. appears



In the downwards direction, it stops at some point, but in the upwards direction it continues ad-infinitum.

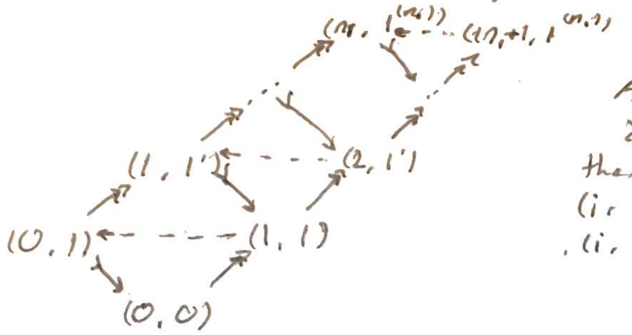
AR-component is $\mathbb{Z}A_\infty/G$, where $A_\infty = 1 \to 2 \to 3 \to \dots$, and $G = (\tau^n)$ for some n or 0. This component is always infinite.



We want to show that this cannot exist.

Assume we have such a component

Obs



Any map $(i, 1^{(r)}) \to X$, $X \notin \{(i, 1^{(r)}), (i+1, 1^{(r+1)}), \dots\}$ then it factors through the mono $(i, 1^{(r)}) \to (i, 1^{(r-1)})$ if $r > 0$, $(i, 1) \to (i, 0)$ if $r = 0$

Pf If $r = n_1$, then it is known (there is only one choice).

Assume it works for $r+1$.

$$\begin{tikzcd}
(i, 1^{(r)}) \arrow[r, tail, "j"] \arrow[d, "f"'] & (i+1, 1^{(r+1)}) \oplus (i, 1^{(r-1)}) \arrow[dl, dashed, "(f_1\ f_2)"] \\
X &
\end{tikzcd}$$

left almost split provided $X \neq (i, 1^{(r)})$

$$\begin{tikzcd}
(i+1, 1^{(r+1)}) \arrow[r, tail] \arrow[d, "f_1"'] & (i+1, 1^{(r)}) \arrow[dl, dashed, "\tilde{f}_1"] \\
X &
\end{tikzcd}$$

Induction hypothesis

note that
$$\begin{tikzcd}
 & (i+1, 1^{(r+1)}) \arrow[dr, tail] & \\
(i, 1^{(r)}) \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & \circlearrowleft & (i+1, 1^{(r)}) \arrow[d, "\tilde{f}_1"] \\
 & (i, 1^{(r-1)}) \arrow[ur] \arrow[r, "\hat{f}_1"'] & X
\end{tikzcd}$$

so $f_1 \circ j_1$ factors as $\hat{f}_1 \circ j_2$ $\square$

Pick M in our component of smallest possible dimension.
As projectives are not in our component, $\mathrm{Hom}(\mathrm{proj}, M) \subseteq \mathrm{rad}^n$ $\forall n$

$\Rightarrow$ $\mathrm{rad}^n(-, M) \neq \mathrm{rad}^{n+1}(-, M)$ $\forall n$
$\Rightarrow$ There is some X_n indec. $X_n \to M \in \mathrm{rad}^n(X, M) \setminus \mathrm{rad}^{n+1}(X, M)$, and X_n is necessarily a summand of $\Theta^n M$.

WLOG $M \simeq (0, 1^{(n_1)})$, so X_n is at $(t, ?)$, where $2t \approx -n$
$|2t + n| \leq 2\Delta$, where $\Delta = \max(\{n_1, n_2, n_3\}) + 1$

If $n \geq 2n_1$, then X_n is at $(t, ?)$, wher $? \neq 0$, then one map factors through the irreducible mono $(t, s^{(r)}) \to (t, s^{(r-1)})$. We can repeat this argument until we are in the τ-orbit $\{(*, 0)\}$

$$\begin{tikzcd} X^n \arrow[rr] \arrow[dr] & & M \\ & (t,0) \arrow[ur] & \end{tikzcd}$$

$t \simeq \frac{n}{2}$, $|2t+n| \leq 3\Delta$

There is a sequence $\{t < 0 \mid \exists$ map $(t,0) \to M$, not in arbitrary powers of the radical$\}$

This never has gaps bigger than 6Δ.