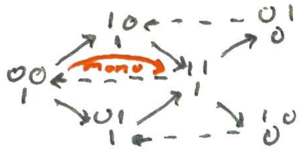
Repteori – Week 12

Exercise 1

a) Find indec. M and a proper mono $\tau M \to M$

$\mathbb{K}\{\cdot \nwarrow \cdot \nearrow \cdot\}$

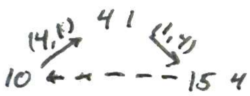


b) Look at the square above.

c) Find $0 \to \tau M \to E \oplus F \xrightarrow{(g_1, g_2)} M$ such that g_1 is mono and E is decomposable.

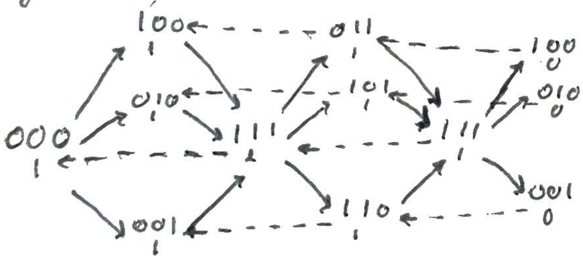
d) Find an example of an almost split seq. w/ 4 or more summands in the middle.

Try $\mathbb{K}\{1 \equiv 2\}$



$\rightsquigarrow \quad 0 \to 10 \to (41)^4 \to 15\,4 \to 0$

Try $\mathbb{K}\{\overset{1}{\cdot} \nwarrow \overset{2}{\underset{4}{\downarrow}} \nearrow \overset{3}{\cdot}\}$



The first square gives an example to this.

Exercise 2 Consider the algebra $\mathbb{K}\{1 \rightrightarrows 2\}$

a) Describe all preprojective and preinjective modules.

Preproj: $P_1 = 10$, $P_2 = 21$

$$\begin{tikzcd}[column sep=small]
 & 21 \arrow[dr, "2"] & & 43 \arrow[ll, dashed] \arrow[dr] & & 65 \arrow[ll, dashed] \arrow[dr] & \cdots \\
10 \arrow[ur, "2"] & & 32 \arrow[ll, dashed] \arrow[ur] & & 54 \arrow[ll, dashed] \arrow[ur] & &
\end{tikzcd}$$

Every preproj have then dim. $n{+}1\ n$. This can be shown with induction.

~~To get~~ For the preinj we get $n\ n{+}1$ by symmetry.

regular: We guess nn.

$$\mathbb{K}^n \underset{J}{\overset{id}{\rightrightarrows}} \mathbb{K}^n$$ This is indec. and therefor regular.

Jordan block form

$$\Phi = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \qquad \Phi^n = \begin{pmatrix} -2n+1 & 2n \\ -2n & 2n+1 \end{pmatrix}$$

Hvis ab er regulær så er $\Phi^n \binom{a}{b} \geq 0 \ \forall n$

$$\Phi^n \binom{a}{b} = \begin{pmatrix} (-2n+1)a + 2n b \\ -2n a + (2n+1) b \end{pmatrix} = \begin{pmatrix} 2n(b-a) + a \\ 2n(b-a) + b \end{pmatrix} \geq 0$$

Hvis $a > b$ så går dette ikke.
Hvis $b < a$ så vil $2n(b-a) < 0$ for $n < 0$. Da finnes n slik at dette ikke går. Derfor må $a = b$.

Assume regular component is of the shape $\mathbb{Z}\{0 \rightarrow 1 \rightarrow \cdots,\ 0 \rightarrow 2 \rightarrow \cdots,\ 0 \rightarrow 3 \rightarrow \cdots\}$

M is our component with minimal dimension.

$X = \{t \mid \exists \text{map } (t,0) \rightarrow M \text{ not in } \operatorname{rad}^n \text{ for } n \gg 0\}$ has size of ~~gaps~~ bounded by some number.

Note that the maps $(t,0) \rightarrow M$ not in arbitrary rad^n must be epi. The image is in the same component, so the image is M.

Apply τ^- (t times), τ^- preserve epis as $\tau^- = \operatorname{Ext}^1(DA, -)$ is right exact.

We get epis $(0,0) \to (\tau^-)^t M \quad \forall t \in X$.

$\Rightarrow \{\dim \tau^{-t} M \mid t \in X\}$ is bounded. Similarly there is some set $Y \subseteq \mathbb{N}$ s.t. Y has bounded gap-size and $\{\dim \tau^t M \mid t \in Y\}$ is bounded

$\Rightarrow$ The dimension of all the indec. in our component is bounded.

Harada - Sai: $\exists n$ s.t. the composition of n indec. maps between indec. modules in our component is 0.

Thus for any X in the component $\Theta^n X \to X$ is 0, i.e. $\mathrm{rad}^n(-, X) = 0$

Thus there are no maps from projectives to X ↯

Thm Any regular component of $\mathrm{mod}_{\mathbb{K}Q}$ is either $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/(\tau^n)$ for some n.

Pf. We have seen these two possibilities are possible. The final possibility cannot exist by the above argument.

Obs When is $\Phi f = f$? This happens iff $-C_A^T C_A^{-1} f = f$ iff $(C_A^T + C_A^{-1}) f = 0$ iff f is an additive function for the underlying graph.

Prop. If Q is connected and not Euclidean, then all reg. comp. are on the form $\mathbb{Z}A_\infty$

Pf. If not, then $\exists M, n \geq 0$ s.t. $\tau^n M = M$. Let $N = \bigoplus_{i=0}^{n-1} \tau^i M$, then $\tau N \simeq N$.

$\Phi\, \underline{\dim}\, N = \underline{\dim}\, N$, then $\underline{\dim}\, N$ is an additive function $\Rightarrow Q$ is Euclidean ↯

Let us assume that Q is Euclidean. Then we have an additive function f. There is some n s.t. $\phi^n = id$ up to some multiple of f.

Prop

M is an indec. KQ module with Q Euclidean.

$\phi^n \underline{\dim} M = \underline{\dim} M + \alpha f$. Then $\alpha = 0$ iff M is reg, $\alpha > 0$ iff M is preinj, $\alpha < 0$ iff M is preproj.

Pf

Recall $\phi f = f$, then $\phi^{2n} \underline{\dim} M = \underline{\dim} M + 2\alpha f$, $\phi^{rn} \underline{\dim} M = \underline{\dim} M + r\alpha f$

M preproj $\Rightarrow$ $\tau^- M$ gives increase in dim, so $\alpha < 0$. If $\alpha < 0$, then $\phi^i \underline{\dim} M < 0$ for some $i \gg 0$.

The case for preinj. is analogous. The case for reg. is the only possibility left. $\square$