Repteori - Uke 13

Was not there for the exercises.

Tame and Wild representation type

Idea for tame algebras: one may in principle find every indec. This is not possible for wild algebras.

Def. $\mathbb{K}$ is alg. closed

An algebra A is tame if for any given dim. vector there are finitely many 1-parameter family of indec. modules and additionally finitely many indec. modules.

Example

$\mathbb{K}\{1 \rightleftarrows 2\}$ dim $= 2$

$\mathbb{K} \underset{a}{\overset{id}{\leftleftarrows}} \mathbb{K}$ 1-param.

$\mathbb{K} \underset{b}{\overset{a}{\leftleftarrows}} \mathbb{K}$ 2-param.

Def.

A functor $F: \mathrm{mod}^A \to \mathrm{mod}^B$ is a rep. embedding if:

- F is faithfull
- F preserves indecomposable
- F reflects isomorphisms

"This functor might be given by a tensor product"

Def.

An algebra A is wild if for any algebra B, ther is a representation embedding $\mathrm{mod}^B \to \mathrm{mod}^A$.

Thm (Drozd)

Any algebra is precisely one of wild and tame.

Pf: "I don't know the proof, but some people say that once you get into it, it is not that bad". □

Fact

$\mathbb{K}Q$ is tame $\Leftrightarrow$ Q is Dynkin or Euclidean

$\rightsquigarrow$ $\mathbb{K}Q$ is wild $\Leftrightarrow$ Q is neither Dynkin nor Euclidean up to connectedness.

Ex

$\mathbb{K}\{1 \substack{\to\\ \to\\ \to} 2\}$ is wild.

This is in fact strictly wild, i.e. F is fully faithfull.
Notice that fully faithfull takes care of preserving indecomposablenes and reflection of isos.

"No local algebra can be strictly wild"

B an arbitrary f.d. algebra, then $B \simeq \mathbb{K}\langle x_1, \dots, x_n\rangle / I$ is a quotient of a free algebra.

free-alg.

Ideal

$\text{mod}^B \xhookrightarrow{\pi^*} \text{f.d.-mod}^{\mathbb{K}\langle x_1,\dots,x_n\rangle}$, where $\pi : \mathbb{K}\langle x_1, \dots, x_n\rangle \twoheadrightarrow B$.

We want a functor $\text{f.d.-mod}^{\mathbb{K}\langle x_1,\dots,x_n\rangle} \xrightarrow{\Phi} \text{mod}^{\mathbb{K}\{1 \substack{\to\\ \to\\ \to} 2\}}$

$$\begin{tikzcd} M \arrow[r, mapsto] & M^{2n+1} \arrow[r, bend left, "id"] \arrow[r, "J"] \arrow[r, bend right, "L"'] & M^{2n+1} \end{tikzcd}$$

where $J = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$ and $L = \begin{pmatrix} 0 & & & & & 0 \\ 1 & \ddots & & & & \\ & \ddots & 0 & & & \\ & & 1 & 0 & & \\ & & & x_1 & \ddots & \\ 0 & & & & x_n & 0 \end{pmatrix}$

Φ is a functor. $f : M \to N$ turns into $\Phi(f) : \Phi(M) \to \Phi(N)$ by applying f in every dimension. $\Phi(f)$ is well-defined.

Φ is faithfull by inspection. Fullness is the difficult part.

Pick a morphism $f : \Phi(M) \to \Phi(N)$.

$f \simeq (A \quad B)$ where $A, B \in \text{Hom}_{\mathbb{K}}(M, N)^{(2n+1)\times(2n+1)}$

By using the three arrows we know that:

- $id \circ A = B \circ id \Rightarrow A = B$
- $JA = AJ$, JA is A moved one step up
  AJ is A moved one step right

$$\Rightarrow A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{2n+1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_2 \\ 0 & & & a_1 \end{pmatrix}$$

- $LA = AL$, LA is A moved one step down with some spices
  AL is A moved one step left with some spices

For $n=3$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & x_1a_1 & x_1a_2 & x_1a_3 & x_1a_4 \\ 0 & 0 & 0 & 0 & x_2a_1 & x_2a_2 & x_2a_3 \\ 0 & 0 & 0 & 0 & 0 & x_3a_1 & x_3a_2 \end{pmatrix}.$$

$$= \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ 0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & a_1x_1 & a_2x_1 & a_3x_1 & 0 \\ 0 & 0 & 0 & 0 & a_1x_2 & a_2x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1x_3 & 0 \end{pmatrix}$$

$\Rightarrow$ $a_i = 0$ if $i \neq 1$ and a_1 comutes with x_i for any i.

Thus $a_1 : M \to N$ is a morphism in f.d. $\mathrm{mod}^{\mathbb{K}\langle x_1,\dots,x_n\rangle}$,
so Φ is surjective on every homset and is full.

Therefore $\mathrm{mod}^{\mathbb{K}\langle x_1,\dots,x_n\rangle} \hookrightarrow \mathrm{mod}^{\mathbb{K}\langle t_1,t_2\rangle}$ is f.f.

Repteori - Week 12

Algebras w/ $\mathrm{rad}^2 = 0$

Let A be an algebra where $\mathrm{rad}^2 A = 0$.
Set B to be the algebra:

$$\begin{pmatrix} A/\mathrm{rad}A & 0 \\ \mathrm{rad}A & A/\mathrm{rad}A \end{pmatrix}$$

$\mathrm{rad}A$ is an $A/\mathrm{rad}A$-bimodule because $\mathrm{rad}^2 A = 0$.

A B-module is a triple (U, V, φ) where U, V are $A/\mathrm{rad}A$-module and $\varphi: V \otimes_{A/\mathrm{rad}A} \mathrm{rad}A \to U$ is $A/\mathrm{rad}A$-linear. φ is mostly implied and omitted

Fact

B is hereditary.

$\rightsquigarrow$ Indec. projectives are:

- $(P/\mathrm{rad}P \quad 0)$ where P is an indec. proj. A-module.
- $(\mathrm{rad}P \quad P/\mathrm{rad}P)$ where P is an indec. proj. A-module.

In particular, all modules $(X \quad 0)$ are proj. since $A/\mathrm{rad}A$ is s.s. So submodules of $(P/\mathrm{rad}P \quad 0)$ are proj.

The radical $\mathrm{rad}(\mathrm{rad}P \quad P/\mathrm{rad}P) = (\mathrm{rad}P \quad 0)$
Remember that $P/\mathrm{rad}P$ is simple and φ maps $\mathrm{rad}P$ to $P/\mathrm{rad}P$ surjectively. Thus this is the radical.
This radical is also projective.

Construction

Let $\phi: \mathrm{mod}^A \to \mathrm{mod}^B$ be a functor acting on mod^A as:

$$M \longmapsto (\mathrm{rad}M, M/\mathrm{rad}M, \varphi) \text{ where}$$

This is module multiplication.

$$\varphi: M/\mathrm{rad}M \otimes_{A/\mathrm{rad}A} \mathrm{rad}A \to \mathrm{rad}M \quad (M \otimes_A \mathrm{rad}A \longrightarrow M\,\mathrm{rad}A)$$

Proposition

Φ is full

Pf.

Let $M, N \in \mathrm{mod}\,\Lambda$, and let $f : \Phi(M) \to \Phi(N)$. Then f is given as a pair (f_1, f_2) s.t.

$f_1 : \mathrm{rad}\,M \to \mathrm{rad}\,N$ and $f_2 : M/\mathrm{rad}\,M \to N/\mathrm{rad}\,N$ where

$$\begin{tikzcd}
M/\mathrm{rad}\,M \otimes_{\Lambda/\mathrm{rad}\,\Lambda} \mathrm{rad}\,\Lambda \arrow[r, two heads, "\varphi_M"] \arrow[d, "f_2 \otimes \mathrm{id}"] & \mathrm{rad}\,M \arrow[d, "f_1"] \\
N/\mathrm{rad}\,N \otimes_{\Lambda/\mathrm{rad}\,\Lambda} \mathrm{rad}\,\Lambda \arrow[r, two heads, "\varphi_N"] & \mathrm{rad}\,N
\end{tikzcd}$$

f_1 is uniquely determined by f_2.

Case 1 - M is proj.

$$\begin{tikzcd}
M \arrow[r, two heads] \arrow[d, dashed, "\exists g"'] & M/\mathrm{rad}\,M \arrow[d, "f_2"] \\
N \arrow[r, two heads] & N/\mathrm{rad}\,N
\end{tikzcd}$$

$\Phi(g)_2 = f_2$, so $\Phi(g) = f$. ✓

Case 2 - M is arbitrary

Recall that q induces an iso from $P/\mathrm{rad}\,P \xrightarrow{\sim} M/\mathrm{rad}\,M$.

Let $P \overset{q}{\twoheadrightarrow} M$ be a proj. cover.

$$\begin{tikzcd}
\Phi(P) \arrow[d, "\Phi(q)"'] \arrow[dr, "f \circ \Phi(q)"] & \\
\Phi(M) \arrow[r, "f"'] & \Phi(N)
\end{tikzcd}$$

Since Φ is full when starting on projectives, $\exists g$ s.t. $\Phi(g) = f \circ \Phi(q)$.

$$\begin{tikzcd}
\mathrm{rad}\,P \arrow[r, two heads, "\Phi(q)_1"] \arrow[d, tail, "j"'] & \mathrm{rad}\,M \arrow[d, tail] \arrow[ddr, bend left, "i \circ f_1"] & \\
P \arrow[r, two heads, "q"] \arrow[d, two heads] \arrow[drr, bend right, "g"'] & M \arrow[d, two heads] \arrow[dr, dashed, "h"] & \\
P/\mathrm{rad}\,P \arrow[r, "\sim"'] & M/\mathrm{rad}\,M & N
\end{tikzcd}$$

This square is exact by char. of PO an PB.

$$g \circ j = i \circ \Phi(g)_1$$
$$= i \circ f_1 \circ \Phi(q)_1$$

Since the square commutes, h exists.

$\Phi(h)_2 \circ \Phi(q)_2 = \Phi(g)_2 = f_2 \circ \Phi(q)_2$

Since q is proj. cov. we know that $\Phi(h)_2 = f_2$ and so $\Phi(h) = f$. $\square$

Proposition

Let $f: M \to N$ in mod^A, then $\Phi(f) = 0$ iff f factors through the inclusion $\mathrm{rad} N \hookrightarrow N$.

Pf.

$\Phi(f) = 0 \Leftrightarrow \Phi(f)_0 = 0$

$$\begin{tikzcd}
M \arrow[r, two heads] \arrow[d, "f"] \arrow[dd, dashed, bend right] \arrow[rd, phantom, "\circlearrowright"] & M/\mathrm{rad} M \arrow[d, "\Phi(f)_0"] \\
N \arrow[r, two heads] & N/\mathrm{rad} N \\
\mathrm{rad} N \arrow[u, hook] \arrow[ru, "0"'] \arrow[u, phantom, shift left=6ex, "\circlearrowright"] &
\end{tikzcd}$$

$\square$

Proposition

Let $(U, V, \varphi) \in \mathrm{mod}^{\otimes}$, then $(U, V, \varphi) \in \mathrm{Im}\, \Phi$ $\Leftrightarrow$ φ is epi.

Pf.

"$\Rightarrow$" This follows from construction.

"$\Leftarrow$" Let $P \twoheadrightarrow V$ be a proj. cov. as A-modules.

$P/\mathrm{rad} P \cong V$, $\mathrm{rad} P = P \,\mathrm{rad} A \cong P \otimes_A \mathrm{rad} A = V \otimes_{A/\mathrm{rad} A} \mathrm{rad} A$

With this we get that $\Phi(P) = (V \otimes_{A/\mathrm{rad} A} \mathrm{rad} A, V, \mathrm{id}_{V \otimes_{A/\mathrm{rad} A} \mathrm{rad} A})$

$$\begin{tikzcd}
V \otimes_{A/\mathrm{rad} A} \mathrm{rad} A \arrow[r, two heads, "\varphi"] \arrow[d, tail] & U \arrow[d, tail] \\
P \arrow[r, two heads] \arrow[d, two heads] \arrow[ru, phantom, "\lrcorner", very near end] & M \arrow[d, two heads] \\
V \arrow[r, "\mathrm{id}_V"'] & V
\end{tikzcd}$$

Since φ is epi and $\mathrm{rad} P \hookrightarrow P$ is mono, the square is in fact exact.

The square tells us that U is a radical as $V \otimes_{A/\mathrm{rad} A} \mathrm{rad} A$ is a radical. Thus

$U = \mathrm{rad} M$ and $V = M/\mathrm{rad} M$.

$\Phi(M) = (U, V, \varphi)$. $\square$