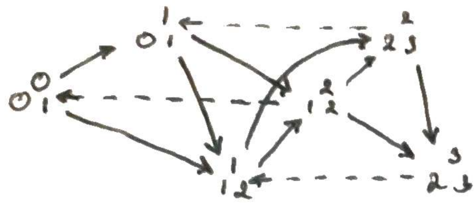
Repteori - Week 14

We discussed oral exam dates. It will be on the 1st and 14th of June.

Exercises

- $\begin{tikzcd}[column sep=small] & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ll] \arrow[ul] \end{tikzcd}$ Find the AR-quiver

preproj $P_1 = \begin{smallmatrix} & 1 & \\ 1 & & 2 \end{smallmatrix}$, $P_2 = \begin{smallmatrix} & 1 & \\ 0 & & 1 \end{smallmatrix}$ and $P_3 = \begin{smallmatrix} & 0 & \\ 0 & & 1 \end{smallmatrix}$



$\ldots \rightsquigarrow \begin{smallmatrix} & n & \\ n & & n+1 \end{smallmatrix}$ or $\begin{smallmatrix} & n+1 & \\ n & & n+1 \end{smallmatrix}$

preinj

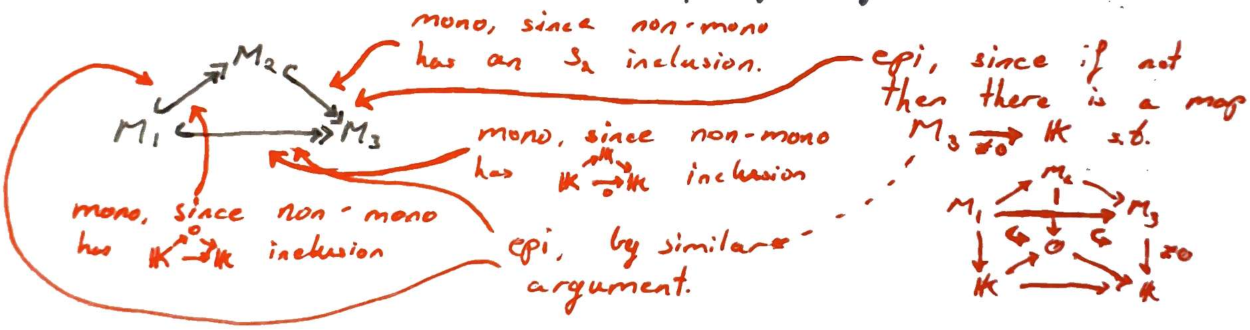
$\begin{smallmatrix} & n & \\ n+1 & & n \end{smallmatrix}$ or $\begin{smallmatrix} & n+1 & \\ n+1 & & n \end{smallmatrix}$

Quasi-simples

- S_2
- $\begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr] & & \mathbb{K} \end{tikzcd}$
- $\begin{tikzcd}[column sep=small] & \mathbb{K} \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr, "\lambda"'] & & \mathbb{K} \end{tikzcd}$
- $\begin{tikzcd}[column sep=small] & \mathbb{K} \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr, "0"'] & & \mathbb{K} \end{tikzcd}$

Are these all of them?

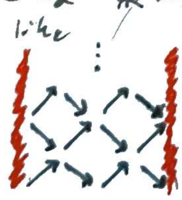
Pick a $\underbrace{\text{module}}^{\text{regular}}$ M. It is indec. q. simple if:



Thus $\begin{tikzcd}[column sep=small] & M_1 \arrow[dr, "id"] & \\ M_1 \arrow[ur, "id"] \arrow[rr, "J"'] & & M_1 \end{tikzcd} = M$ where $J = J_n(\lambda)$ is invertible, are regular indec.

Finding components:

$\tau S_2 = \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr] & & \mathbb{K} \end{tikzcd}$ and $\tau \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr] & & \mathbb{K} \end{tikzcd} = S_2$, thus the component looks like



These sides are glued to each other.

Trick: Look for non-trivial extensions and use the AR-formula.

$$\tau\left(\begin{tikzcd}[column sep=small] & \mathbb{K} \arrow[dr, "id"] & \\ \mathbb{K} \arrow[ur, "id"] \arrow[rr, "\lambda"'] & & \mathbb{K}\end{tikzcd}\right):$$

$$\mathbb{K}\left\{\begin{tikzcd}[column sep=small] & 2 \arrow[dl, "a"'] & \\ 1 & & 3 \arrow[ul, "b"'] \arrow[ll, "c"]\end{tikzcd}\right\}$$

$$\begin{tikzcd}[column sep=small] & \mathbb{K} \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr, "\lambda"'] & & \mathbb{K}\end{tikzcd} \twoheadleftarrow P_1 \xleftarrow{c-\lambda ab} P_3, \qquad (1\ -\lambda) \mapsto c-\lambda ab$$

$$P_1 = \begin{tikzcd}[column sep=small] & \mathbb{K}a \arrow[dr] & \\ \mathbb{K}e_1 \arrow[ur] \arrow[rr] & & \mathbb{K}c \oplus \mathbb{K}ab\end{tikzcd}$$

$$P_3 = \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & \mathbb{K}e_3\end{tikzcd}$$

$$i_1 = \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ \mathbb{K}e_1^* \arrow[ur] \arrow[rr] & & 0\end{tikzcd}$$

$$\uparrow\ c-\lambda ab \mapsfrom (1\ -\lambda)$$

Think of this as some tensored thing.

$$i_3 = \begin{tikzcd}[column sep=small] & \mathbb{K}b^* \arrow[dr] & \\ \mathbb{K}c^* \oplus \mathbb{K}(ab)^* \arrow[ur] \arrow[rr] & & \mathbb{K}e_3^*\end{tikzcd}$$

$$\uparrow\ 1 \mapsto \lambda c^* + (ab)^*$$

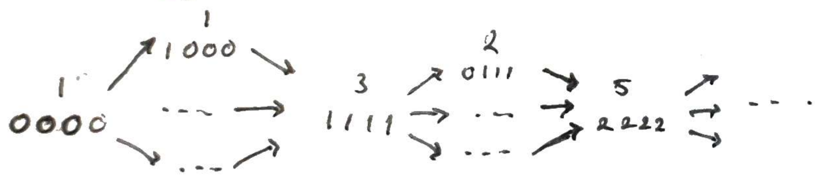
$$\begin{tikzcd}[column sep=small] & \mathbb{K}b^* \arrow[dr, "1"] & \\ \mathbb{K} \arrow[ur, "1"] \arrow[rr, "\lambda"'] & & \mathbb{K}e_3^*\end{tikzcd}$$

This map satisfies being a kernel.

so $\tau\ \begin{tikzcd}[column sep=small] & \mathbb{K} \arrow[dr] & \\ \mathbb{K} \arrow[ur] \arrow[rr, "\lambda"'] & & \mathbb{K}\end{tikzcd}$

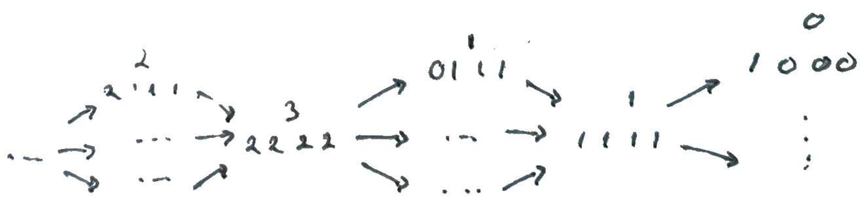
- $$\begin{tikzcd}[column sep=small] & & 0 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\ 1 & 2 & 3 & 4\end{tikzcd}$$

preproj



$\left(\begin{smallmatrix} & 2n+1 & \\ n & n\ n & n\end{smallmatrix}\right), \left(\begin{smallmatrix} & 2n+1 & \\ n+1 & n\ n & n\end{smallmatrix}\right), \left(\begin{smallmatrix} & 2n & \\ n-1 & n\ n & n\end{smallmatrix}\right)$ up to permutation

pre inj



$\left(\begin{smallmatrix} & 2n+1 & \\ n+1 & n+1\ n+1 & n+1\end{smallmatrix}\right), \left(\begin{smallmatrix} & 2n & \\ n+1 & n\ n & n\end{smallmatrix}\right), \left(\begin{smallmatrix} & 2n & \\ n & n+1\ n+1 & n+1\end{smallmatrix}\right)$

Guessing q-simple regular

- $\begin{pmatrix} & 0 & \\ 1 & 1 & 0 & 0 \end{pmatrix}$ up to permutation $\leftsquigarrow$ M_{12}

Let $X = $
$$\begin{tikzcd} & & X_0 & & \\ X_1 \arrow[urr, hook] & X_2 \arrow[ur, hook] & & X_3 \arrow[ul, hook] & X_4 \arrow[ull, hook] \end{tikzcd}$$

$\mathrm{Hom}(M_{12}, X) = X_1 \cap X_2$, $\mathrm{Hom}(M_{ij}, X) = 0 \ \forall i,j$
$\Leftrightarrow$ $X_1, \dots, X_4$ are pairwise disjoint.

$\mathrm{Hom}(X, M_{12}) = \{ \varphi : X_0 \to \mathbb{K} \mid \text{s.t. } \varphi|_{X_3 + X_4} = 0 \}$

$\rightsquigarrow$
$$\begin{tikzcd} & & X_0 \arrow[d, dashed] & & \\ X_1 \arrow[urr, hook] \arrow[dd] & X_2 \arrow[ur, hook] \arrow[dd] & \mathbb{K} & X_3 \arrow[ul, hook] & X_4 \arrow[ull, hook] \\ & & & & \\ \mathbb{K} \arrow[uurr, "id"] & \mathbb{K} \arrow[uur, "id"] & & 0 \arrow[uul] & 0 \arrow[uull] \end{tikzcd}$$

$\mathrm{Hom}(X, M_{ij}) = 0 \ \forall i,j$
$\Leftrightarrow$ $X_1, \dots, X_4$ pairwise span X_0

X_0 is the direct sum of any two of $X_1, \dots, X_4$
$\rightsquigarrow$ $X_1 \simeq X_2 \simeq X_3 \simeq X_4 \simeq \mathbb{K}^n$ Important, this is not equality, but just an isomorphism.

$$\begin{tikzcd} & & \mathbb{K}^{2n} & & \\ \mathbb{K}^n \arrow[urr, "\binom{1}{0}"] & \mathbb{K}^n \arrow[ur, "\binom{0}{1}"] & & \mathbb{K}^n \arrow[ul, "\binom{A}{B}"'] & \mathbb{K}^n \arrow[ull, "\binom{C}{D}"'] \end{tikzcd}$$

If A is not epi, then $X_2 + X_3 \neq X_0$, so A is an iso as it is also square.
$\rightsquigarrow$ similarly for B, C and D.

wlog assume $\binom{A}{B} = \binom{1}{1}$, by fixing the basis given by A and B for the $\mathbb{K}$s.

The same allows us to let C be given by 1 and D is something similar. Let D be a Jordan block form.

$\rightsquigarrow$
$$\begin{tikzcd} & & \mathbb{K} & & \\ \mathbb{K} \arrow[urr, "\binom{1}{0}"] & \mathbb{K} \arrow[ur, "\binom{0}{1}"] & & \mathbb{K} \arrow[ul, "\binom{1}{1}"'] & \mathbb{K} \arrow[ull, "\binom{1}{\lambda}"'] \end{tikzcd}$$
for $\lambda \in \mathbb{K} / \{0, 1\}$

These are the $\smile$ q-simple.

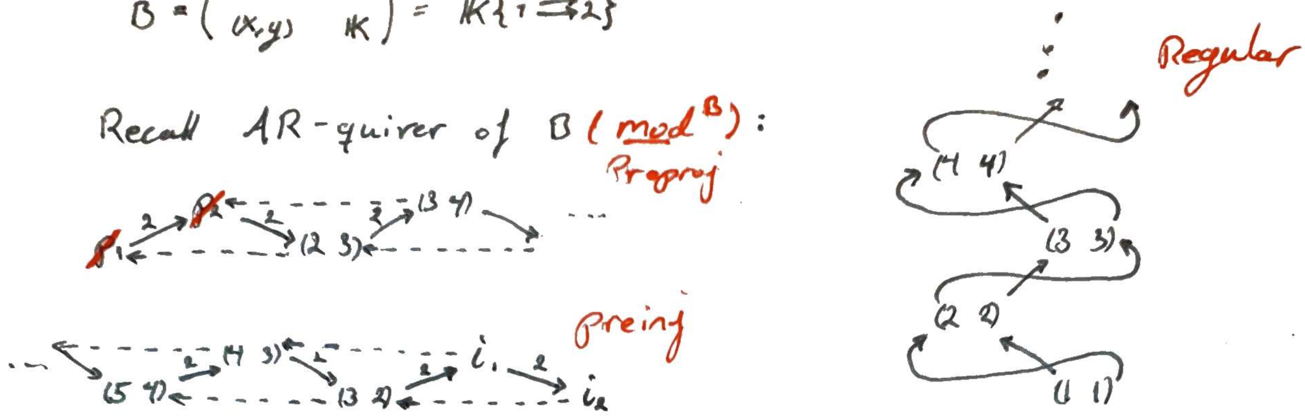
Claim:

$\tau(M_\lambda) = M_\lambda$, $\tau(M_{12}) = M_{34}$

- Let $A = \mathbb{K}[x,y]/(x^2, y^2, xy)$, find B and the AR-quiver of A.

$$B = \begin{pmatrix} \mathbb{K} & 0 \\ (x,y) & \mathbb{K} \end{pmatrix} = \mathbb{K}\{1 \rightrightarrows 2\}$$

Recall AR-quiver of B ($\underline{\mathrm{mod}}^B$):



We have the functor $\phi : \mathrm{mod}^A \to \mathrm{mod}^B$
$M \mapsto (\mathrm{rad}\, M,\ M/\mathrm{rad}\, M)$.

Image(ϕ) = everything except P_1

Pick a B-module $(M_2\ \ M_1)$, then we have $\varphi : M_1 \otimes \mathrm{rad}\, A \to M_2$ which is the action of the arrows.

For two M, N A-modules which are indec. then $M \not\cong N \Rightarrow \phi M \not\cong \phi N$

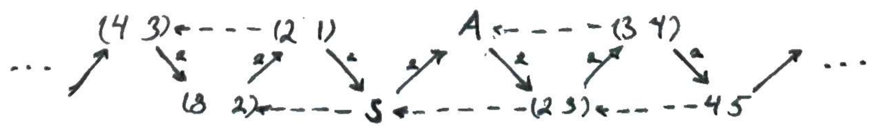
The projectivo for A are A. $\mathrm{rad}\, A = \mathbb{K}x \oplus \mathbb{K}y \simeq s^2$.
We have every arrow into A, and 2 out of s.

$\tau^- s$:

$$\begin{array}{c} s \rightarrowtail i \to i^2 \\ \big\downarrow \nu^- \\ A \to A^2 \twoheadrightarrow \tau^- s \end{array}$$

$\tau^- s / \mathrm{rad}\, \tau^- s$ is 2 dim. and $\mathrm{rad}\, \tau^- s$ is 3 dim.

The AR-quiver of A:



The regular component is unchanged

Construction

Let Q be a quiver. The seperated quiver of Q is given by: vertices $= \{i, i' \mid i$ vertex of $Q\}$ and arrows $= \{i \xrightarrow{\alpha} j' \mid \alpha : i \to j$ in $Q\}$

Ex

$$\begin{tikzcd} 1 \arrow[r] & 2 \end{tikzcd} \rightsquigarrow \begin{tikzcd} 1 \arrow[dr] & 2 \\ 1' & 2' \end{tikzcd}$$

Lemma

$A = \Bbbk Q/(\text{arrows}^2)$, $B = \begin{pmatrix} A/\operatorname{rad} A & 0 \\ \operatorname{rad} A & A/\operatorname{rad} A \end{pmatrix}$. then

$B \cong \Bbbk(\text{seperated arrows of } Q)$

Pf

vertices of quiver of $B \triangleq$ simple B-modules $= \{(S \;\; 0), (0 \;\; S) \mid S$ simple A-module$\}$

arrows of quiver of $B \simeq \operatorname{rad} B/\operatorname{rad}^2 B \cong \begin{pmatrix} 0 & 0 \\ \operatorname{rad} A & 0 \end{pmatrix}$

$\simeq$ arrows of A

However, right multiplication on arrows is restricted from $(0 \;\; X)$ to $(Y \;\; 0)$. $\square$

Corollary

$\Bbbk Q/(\text{arrows}^2)$ is rep. finite iff the seperated quiver of Q is a disjoint union of Dynkin quivers.

Notes for the exam

- Mads is the "sensor"
- You should expect to find an AR-quiver

Recap of the course

- $\nu : \text{proj}\, A \xrightarrow{\simeq} \text{inj}\, A \rightsquigarrow - \otimes DA$

  $D\mathrm{Hom}(P, M) \simeq \mathrm{Hom}(M, \nu P)$ functorially in both args. arguments.

  If gl. dim. $< \infty$:

  left-derive

  $\mathbb{L}\nu$ is called a "Serre functor"

$$\begin{array}{ccc} \mathbb{L}\nu : D^b(\text{mod}^A) & \longrightarrow & D^b(\text{mod}^A) \\ \uparrow \wr & & \uparrow \wr \\ \nu : K^b(\text{proj}\, A) & \longrightarrow & K^b(\text{inj}\, A) \end{array}$$

  gl. dim. is finite

  $D\mathrm{Hom}_{dv}(X, Y) \simeq \mathrm{Hom}_{dv}(Y, \mathbb{L}\nu X)$ functorially in both args.

- τ: constructed from ν

  $D\mathrm{Ext}^1(M, N) \simeq \overline{\mathrm{Hom}}(N, \tau M) \simeq \underline{\mathrm{Hom}}(\tau^- N, M)$

  $\tau_{dv} = \mathbb{L}\nu[-1]$, then $D\mathrm{Ext}^1(X, Y) \simeq D\mathrm{Hom}_{dv}(X, Y[1])$
  $\simeq \mathrm{Hom}_{dv}(Y[1], \mathbb{L}\nu X) \simeq \mathrm{Hom}_{dv}(Y, \tau X)$

- There are almost split sequences
  Happel: $D^b(\text{mod}^A)$ has almost split triangles iff gl. dim. $A < \infty$
  We may construct AR-quivers

- "Knitting". If A is connected, then if the AR-quiver have a finite component, then we have everything.

  The Grothendieck group $K_0(D^b(\text{mod}^A))$ has less information than for mod^A. "Knitting" becomes more strenuous in the derived category. There are never finitely many indec. for the derived category.

- Things are nice if A is hereditary. ($A \simeq \mathbb{K}Q$)
  We know the AR-quiver depends on the graph Q.

  X indec in $D^b(\text{mod}^A) \Rightarrow X$ is concentrated in one degree.
  $M \in \text{mod}^A$ not proj. then $\mathbb{L}\nu M$ is

  $$0 \to P_1 \to P_0 \to 0 \overset{\nu}{\mapsto} 0 \to \nu P_1 \twoheadrightarrow \nu P_0 \to 0$$

$\tau_{\mathcal{D}^b} M = \tau M$

$\tau_{\mathcal{D}^b} P = \nu P(-1)$

almost split triangles are those coming from mod^A and

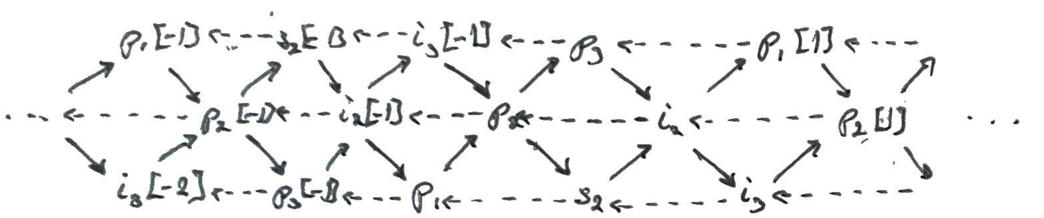
$\tau_{\mathcal{D}^b} P \longrightarrow (\nu P / \mathrm{soc}\, \nu P)[-1] \oplus \mathrm{rad}\, P \longrightarrow P \longrightarrow \tau_{\mathcal{D}^b} P[1]$

Warning: If A is not hereditary then

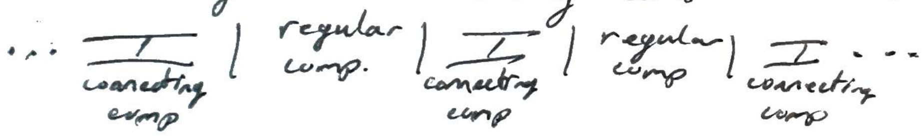
- $\exists$ indec which are not just shifts of modules
- $\tau_{\mathcal{D}^b} M = \tau M$ only for ?

AR-quiver for $\mathcal{D}^b(\mathrm{mod}^{\mathbb{K}Q})$

- Q Dynkin: $\mathbb{Z}Q$. ex: $Q = \{1 \to 2 \to 3\}$



- Q not Dynkin: connecting component $\mathbb{Z}Q$ ← preproj & preinj



Tilting

"Comparing module categories w/ the same derived category"

special case: A is hereditary $\rightsquigarrow$ know the derived category well

can get information about mod^B from $\mathcal{D}^b(\mathrm{mod}^B)$

Richard's theorem gives an explicit condition for when this happens.