Representation theory - Week 2

Fact: M finitely generated, then every proper submodule is contained in a maximal one.

Zorn's lemma: (proper submodules, $\subseteq$) is poset, then every chain has an upper bound: the union.

Then there is a maximal element

Since M is f.g. the union is proper, thus the maximal is proper.

Prop: If M is finitely generated, then $\operatorname{rad} M$ is the biggest superflous submodule.

Def: A submodule $N \subseteq M$ is superflous if $\forall H \subseteq M \mid N + H = M$ only if $H = M$. Also small

Pf.
Assume that H is maximal. Then $H + \operatorname{rad} M = H$. Thus given $H \subseteq M$ s.t. $H + \operatorname{rad} M = M$, then $H' \subseteq H$ where H' is maximal for any H', so $H = M$. $\square$

Cor 1 If M is f.g., then $\operatorname{rad} M \subset M$, or $M = 0$.

Cor 2 $P \overset{f}{\twoheadrightarrow} M$ is an epi from a f.g. projective, then f is a projective cover if and only if $\operatorname{Ker} f \subseteq \operatorname{rad} P$.

Cor 3 A artinian ring. Then $\operatorname{rad} A$ is the biggest
- nilpotent ideal
- nil left/right ideal

Pf. Every right ideal is f.g. due to finite length.
Then the radical chain:
$A \supset \operatorname{rad} A \supset (\operatorname{rad} A)^2 \supset (\operatorname{rad} A)^3 \supset \dots$

By artinian, the radical filtration is finite. Thus $(\operatorname{rad} A)^n = 0$ for some $n < \infty$.

If N is a nil right ideal, then $\forall n : N \, \forall a : A \;\; 1 + na$ invertible.

$(1+na)^{-1} = 1 - na + (na)^2 - (na)^3 + (na)^4 - \dots$

$\Rightarrow n : \operatorname{rad} A.$ $\square$

If P indec proj: assume $P/\operatorname{rad} P$ is not necessarily simple

Then $\exists\ P/\operatorname{rad} P \twoheadrightarrow S$ simple

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, dashed, bend right] & S \\ \text{Proj cov} \arrow[u, dashed, bend right] \arrow[ur] & \end{tikzcd}$$

Proj. cov. is now a summand of P, by indecomposableness we get

$P =$ Proj cov and $S = P/\operatorname{rad} P$.

Cor P indec. proj $\Rightarrow$ $\operatorname{rad} P$ is a maximal submodule

Cor A_A indec. $\Leftrightarrow$ $\operatorname{rad} A$ is a maximal right module
$\Leftrightarrow$ A is local

Obs A local $\Leftrightarrow$ $\forall$ non-invertible $a \in A \mid 1+a$ is inv.

Thm (Krull-Schmidt)

Let M be a finite dim. module over some $\mathbb{K}$-algebra. Then M has an $\underbrace{\text{essentially unique}}_{\text{unique up to isomorphism and permutation}}$ decomposition into indecomposables.

Moreover these indecomposables have local endomorphism rings.

Remark

Existence of a decomposition is clear by finite dim.

Lemma 1

M a f.d. module, then M indec if and only if $\operatorname{End}(M)$ is local.

Pf: "$\Leftarrow$" (true without f.d.) Suppose $M = M_1 \oplus M_2$, then there are two idempotents:

$$\begin{tikzcd} M_1 \arrow[r, bend left, "\iota_1"] & M \arrow[l, bend left, "\pi_1"] \arrow[r, bend left, two heads, "\pi_2"] & M_2 \arrow[l, bend left, "\iota_2"] \end{tikzcd}$$

s.t. $\iota_1\pi_1 + \iota_2\pi_2 = id_M$. As $\operatorname{End}(M)$ local, then either $\iota_k\pi_k$ $k \in \{1,2\}$ is invertible. Then if $\iota_1\pi_1$ is invertible. then $\langle \iota_1\pi_1 \rangle = \operatorname{End}(M) \Rightarrow M_1 \cong M$.

"$\Rightarrow$" note that $End(M)$ is a finite dimensional algebra. ($End(M) \subseteq End_{\mathbb{K}}(M) \subseteq M_{\dim M}\mathbb{K}$). $End(M)$ is Artinian.

If $End\, M$ is not lokal, then it admits a proper decomposition. i.e. $End\, M \simeq P \oplus Q$ as right $End\, M$-modules.

$M \simeq End\, M \otimes_{End M} M$ (note: M is naturally an $End\, M - A -$ bimodule)

$\simeq P \otimes_{End M} M \oplus Q \otimes_{End M} M$

Since M is indec. one of the summands are 0, e.g. $P \otimes_{End M} M = 0$.

$$P \otimes_{End M} M \xrightarrow{\iota \otimes id_M} End\, M \otimes M \xrightarrow[\sim]{ap} M$$

$$\varphi \otimes m \longmapsto \varphi \otimes m \longmapsto \varphi(m)$$

If $\varphi \neq 0\ \exists m : M \mid \varphi(m) \neq 0$, i.e. $\varphi \otimes m \neq 0$

$P \otimes M = 0 \quad \lightning \quad P \neq 0 \Rightarrow P = 0$ $\square$

<u>Remark</u>: Statement is also true for M finite length (but needs a new proof).

<u>Lemma 2</u>

$M = \bigoplus_{i=1}^{m} M_i$, where M_i has local endo-ring. and

$M = \bigoplus_{i=1}^{n} N_i$ where N_i are indec.

Then the M_i's and N_i's coincide up to isomorphisms and permutation

<u>Pf</u> Induction on m: $m=1$ is ok by prev. results.

Let $M \simeq M_1 \oplus M_{rest} \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \bigoplus_{i=1}^{n} N_i$ composition of both directions is id_M.

Let $(\alpha_{\cdot\cdot n})$ ... See online notes. $\square$

Now Krull-Schmidt then follows from both of these lemmata.

General setup for this course:

- A is a finite dim. algebra, and we want to understand the category mod^A (modules are representations)
- Krull-Schmidt thm: may focus on indecomposable modules. It might be hard to find decompositions and in geometry they mix w/ decomposables.

Exercise

- Find all indec. (contravariant) representations of

$$Q = \begin{tikzcd} 0 \arrow[d] \\ 1 \end{tikzcd} \qquad Q = \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd} \qquad Q = \begin{tikzcd} & 0 \arrow[dl] \arrow[d] \arrow[dr] & \\ 1 & 2 & 3 \end{tikzcd}$$

- Show that $Q = \begin{tikzcd} & & 0 \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\ 1 & 2 & & 3 & 4 \end{tikzcd}$ has ∞-many indec. representations.

Morita equivalence

Thm Let R, S be rings. When are $\mathrm{mod}R \simeq \mathrm{mod}S$?

$\mathrm{mod}R \simeq \mathrm{mod}S \Leftrightarrow \exists P \in \mathrm{mod}^R$ which is a progenerator, i.e. projective and $\forall M \in \mathrm{mod}^R \ \exists$ epi $p: P^n \twoheadrightarrow M$, s.t. $S \cong \mathrm{End}(P)$.

Pf.

"$\Rightarrow$" Let $F: \mathrm{mod}^S \xrightarrow{\sim} \mathrm{mod}^R$, then let $F(S) = P$.
S is a progenerator $\Rightarrow P = F(S)$ is still a progenerator

$S = \mathrm{End}_S(S) = \mathrm{End}_R(F(S)) = \mathrm{End}_R(P)$ as F is equiv.

"$\Leftarrow$" note: P is an S-R-bimodule. We obtain the adjunction:

$$\begin{tikzcd} \mathrm{mod}^R \arrow[rr, bend right, "\mathrm{Hom}_R(P, -)"'] \arrow[rr, phantom, "\perp"] & & \mathrm{mod}^S \arrow[ll, bend right, "- \otimes_S P"'] \\ \{P^n\} \arrow[u] \arrow[rr, bend right, "\simeq"'] & & \{S^n\} \arrow[u] \arrow[ll, bend right, "\simeq"'] \end{tikzcd}$$

Both are right exact.

restricts to an equivalence

This equivalence propogates to cokernels of maps between P^n's / S^n's.

As P and S are progen, these are all of mod^R and mod^S. $\square$

Ex

- $\mathbb{Z}_{\mathbb{Z}}$ has $\operatorname{rad}\mathbb{Z}_{\mathbb{Z}} = 0$
- $\mathbb{K}[x]$ $\operatorname{rad}\mathbb{K}[x] = \bigcap_{a:\mathbb{K}} \langle x+a \rangle = 0$
- $\mathbb{K}[\![x]\!]$ $\operatorname{rad}\mathbb{K}[\![x]\!] = \langle x \rangle \leftarrow$ Everything w/ non-zero constant is invertible
- $\mathbb{K}Q$ where Q is a finite acyclic quiver
  $\operatorname{rad}\mathbb{K}Q = \langle$ the arrows $\rangle$
- $M_n(\mathbb{K})$ $\operatorname{rad} M_n(\mathbb{K}) = \langle 0 \rangle$
  $M_n(\mathbb{K})$ has no non-trivial two-sided ideals
- $T_n(\mathbb{K})$ upper triangular $n\times n$-matrices over $\mathbb{K}$.
  $\operatorname{rad} T_n \mathbb{K} = \langle$ matrices w/ zero on diagonal $\rangle$
- $\mathbb{K}[x]/(x^2-x)$ $\operatorname{rad}\mathbb{K}[x]/\langle x^2-x\rangle = 0$ as there are no nilpotent elements.

Note that $x^2-x = x(x-1)$ so
$\mathbb{K}[x]/(x^2-x) \simeq \mathbb{K}[x]/(x) \times \mathbb{K}[x]/(x-1)$

Projectives and simples

Let A be an Artinian ring

Prop: The functor $P \mapsto P/\operatorname{rad}P$ gives bijections

$$\text{indec. proj}/\simeq \xrightarrow{\sim} \text{simples}/\simeq$$

$$\text{f.g. proj}/\simeq \xrightarrow{\sim} \text{f.g. s.s.}/\simeq$$

Pf:

$P/\operatorname{rad}P$ is s.s.

Construction in the other direction: Projective cover
(note: this is not a functor)

Know that P is the projective cover of $P/\operatorname{rad}P$.

Let S be simple and P the projective cover.

$$\begin{tikzcd} \operatorname{Ker} p \arrow[r, tail] \arrow[d, hook] & P \arrow[r, two heads, "p"] & S \\ \operatorname{rad} P \arrow[ur] & & \end{tikzcd}$$

$\operatorname{Ker} p$ is max. as S is simple
$\Rightarrow \operatorname{Ker} p \simeq \operatorname{rad} P$.
$\Rightarrow \operatorname{rad} P$ is maximal
$\Rightarrow P$ is indecomposable