

Representation theory - week 2

Fact: M finitely generated, then every proper submodule is contained in a maximal one.

Zorn's lemma: (proper submodules, $\subseteq$) is poset, then every chain has an upper bound: the union.

Then there is a maximal element

Since M is f.g. the union is proper, thus the maximal is proper.

Prop: If M is finitely generated, then $\text{rad} M$ is the biggest superfluous submodule.

Def: A submodule $N \subseteq M$ is superfluous if $\forall H \subseteq M \mid N+H=M$ only if $H=M$. Also small

Pf.

Assume that H is maximal. Then $H + \text{rad} M = H$. Thus given $H \subseteq M$ s.t. $H + \text{rad} M = M$, then $H' \subseteq H$ where H' is maximal for any H' , so $H = M$. $\square$

Cor 1 If M is f.g., then $\text{rad} M \subset M$, or $M = 0$.

Cor 2 $P \xrightarrow{f} M$ is an epi from a projective, then f is a projective cover if and only if $\text{Ker } f \subseteq \text{rad } P$.

Cor 3 A artinian ring. Then $\text{rad } A$ is the
biggest
• nilpotent ideal
• nil left/right ideal

Pf. Every right ideal is f.g. due to finite length.

Then the radical chain:

$$A \supset \text{rad } A \supset (\text{rad } A)^2 \supset (\text{rad } A)^3 \supset \dots$$

By artinian, the radical filtration is finite. Thus $(\text{rad } A)^n = 0$ for some $n < \infty$.

If N is a nil right ideal, then $\forall n: N \forall a: A \mid 1+na$ invertible.

$$(1+na)^{-1} = 1 - na + (na)^2 - (na)^3 + (na)^4 - \dots$$

$$\Rightarrow n \subseteq \text{rad } A. \quad \square$$

If P indec proj: ^{assume} $P/\text{rad } P$ is not necessarily simple
 Then $\exists P/\text{rad } P \twoheadrightarrow S$ simple


 Proj. cov. is now a summand of P , by indecomposability we get
 $P = \underset{\text{cor}}{\text{Proj}}$ and $S = P/\text{rad } P$.

Cor P indec. proj $\Rightarrow \text{rad } P$ is a maximal submodule

Cor A indec. $\Leftrightarrow \text{rad } A$ is a maximal right module
 $\Leftrightarrow A$ is local

Ob A local $\Leftrightarrow \forall$ non-invertible $a: A \mid 1+ra$ is inv.

Thm (Krull-Schmidt)

Let M be a finite dim. module over some K -algebra.

Then M has an essentially unique decomposition into indecomposables.
 unique up to isomorphism and permutation

Moreover these indecomposables have local endomorphism rings.

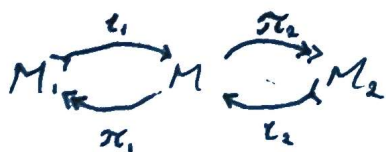
Remark

Existence of a decomposition is clear by finite dim.

Lemma 1

M a f.d. module, then M indec if and only if $\text{End}(M)$ is local.

Pf: (true without f.d.) Suppose $M = M_1 \oplus M_2$, then there are two idempotents:



s.t. $e_1 \pi_1 + e_2 \pi_2 = \text{id}_M$. As $\text{End}(M)$ is local, then either $e_k \pi_k$ $k \in \{1, 2\}$ is invertible. Then if $e_1 \pi_1$ is invertible, then $\langle e_1 \pi_1 \rangle = \text{End}(M) \Rightarrow M_1 = M$.

" $\Rightarrow$ " note that $\text{End}(M)$ is a finite dimensional algebra.
 ($\text{End}(M) \subseteq \text{End}_K(M) \subseteq M_{\dim M}(K)$). $\text{End}(M)$ is Artinian.

If $\text{End } M$ is not local, then it admits a ^{proper} decomposition,
 i.e. $\text{End } M \cong P \oplus Q$ as right $\text{End } M$ -modules.

$$M \cong \text{End } M \otimes_{\text{End } M} M \quad (\text{note: } M \text{ is naturally an } \text{End } M\text{-}A\text{-bimodule})$$

$$\cong (P \otimes_{\text{End } M} M \oplus Q \otimes_{\text{End } M} M)$$

Since M is indec. one of the summands are 0,
 e.g. $P \otimes_{\text{End } M} M = 0$.

$$P \otimes_{\text{End } M} M \xrightarrow{\otimes \text{id}_M} \text{End } M \otimes_{\text{End } M} M \xrightarrow{\cong} M$$

$$Q \otimes_{\text{End } M} M \xrightarrow{\quad} Q \otimes_{\text{End } M} M \xrightarrow{\quad} Q(M)$$

If $Q \neq 0 \exists m: M \mid Q(M) \neq 0$, i.e. $Q \otimes_{\text{End } M} M \neq 0$

$$P \otimes_{\text{End } M} M = 0 \quad \forall P \neq 0 \Rightarrow P = 0 \quad \square$$

Remark: Statement is also true for M finite length
 (but needs a new proof).

Lemma 2

$M = \bigoplus_{i=1}^n M_i$, where M_i has local endo-ring, and

$M = \bigoplus_{i=1}^n N_i$ where N_i are indec.

Then the M_i 's and N_i 's coincide up to isomorphisms and permutation

Pf.

Induction on m : $m=1$ is ok by prev. results.

Let $M \cong M_1 \oplus M_{\text{rest}} \xleftarrow[\alpha]{\beta} \bigoplus_{i=1}^n N_i$ composition of both directions is id $_M$.

Let $(\alpha, \dots, \alpha)$

... See online notes. $\square$

Now Krull-Schmidt then follows from both of these lemmata.

General setup for this course:

- A is a finite dim. algebra, and we want to understand the category $\text{mod } A$ (modules are representations)
- Krull-Schmidt thm: may focus on indecomposable modules. It might be hard to find decompositions and in geometry they mix w/ decomposables.

Exercise

- Find all indec. (contravariant) representations of

$$Q = \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}$$

$$Q = \begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ 1 & & 2 \end{array}$$

$$Q = \begin{array}{ccc} & 0 & \\ \swarrow & \downarrow & \searrow \\ 1 & 2 & 3 \end{array}$$

- Show that $Q = \begin{array}{cccc} & 0 & & \\ \swarrow & \downarrow & \searrow & \\ 1 & 2 & 3 & 4 \end{array}$ has ∞ -many indec. representations.

Morita equivalence

Thm Let R, S be rings. When are $\text{mod } R \cong \text{mod } S$?

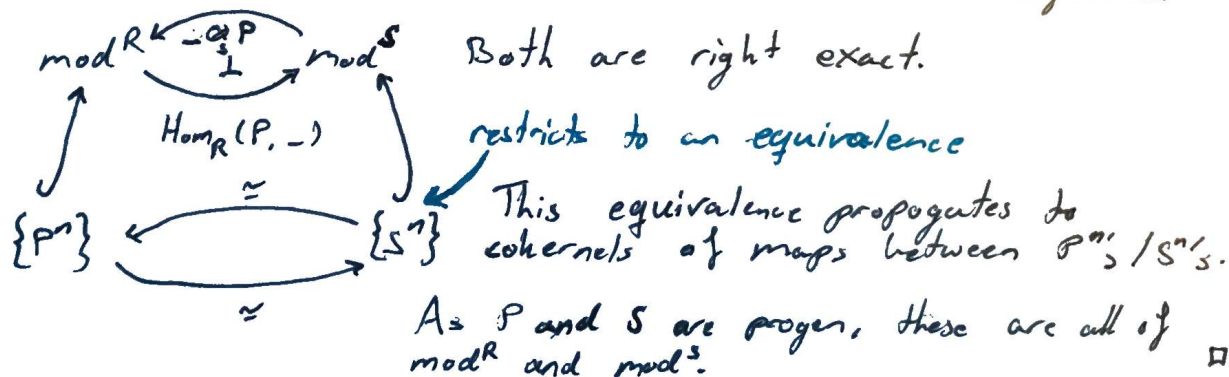
$\text{mod } R \cong \text{mod } S \Leftrightarrow \exists P \in \text{mod } R$ which is a progenerator, i.e. projective and $\forall M \in \text{mod } R \exists \text{ epi } p: P^n \rightarrow M$, s.t. $S = \text{End}(P)$.

Pf.

" $\Rightarrow$ " Let $F: \text{mod } S \xrightarrow{\sim} \text{mod } R$, then let $F(S) = P$.
 S is a progenerator $\Rightarrow P = F(S)$ is still a progenerator

$S = \text{End}_S(S) = \text{End}_R(F(S)) = \text{End}_R(P)$ as F is equiv.

" $\Leftarrow$ " wtc: P is an S - R -bimodule. We obtain the adjunction:



Ex • $\mathbb{Z}_{\mathbb{Z}}$ has $\text{rad } \mathbb{Z}_{\mathbb{Z}} = 0$

• $K[x]$ $\text{rad } K[x] = \bigcap_{a \in K} \langle x+a \rangle = 0$

• $K[[x]]$ $\text{rad } K[[x]] = \langle x \rangle$ ← Everything w/ non-zero constant is invertible

• KQ where Q is a finite acyclic quiver
 $\text{rad } KQ = \langle \text{the arrows} \rangle$

• $M_n(K)$ $\text{rad } M_n(K) = \langle 0 \rangle$

$M_n(K)$ has no non-trivial two-sided ideals

• $T_n(K)$ upper triangular $n \times n$ -matrices over K .
 $\text{rad } T_n(K) = \langle \text{matrices w/ zero on diagonal} \rangle$

• $K[x]/(x^2-x)$ $\text{rad } K[x]/(x^2-x) = 0$ as there are no nilpotent elements.

Note that $x^2-x = x(x-1)$ so
 $K[x]/(x^2-x) \cong K[x]/(x) \times K[x]/(x-1)$

Projectives and simples

Let A be an Artinian ring

Prop: The functor $P \mapsto P/\text{rad } P$ gives bijections

$\text{indec. proj} / \cong \xrightarrow{\sim} \text{simple} / \cong$

f.g. proj / $\cong \xrightarrow{\sim}$ f.g. s.s. / $\cong$

Pf:

$P/\text{rad } P$ is s.s.

Construction in the other direction: Projective cover
 (note: this is not a functor)

Know that P is the projective cover of $P/\text{rad } P$.

Let S be simple and P the projective cover.

$$\begin{array}{c} \text{Ker } p \xrightarrow{\quad} P \xrightarrow{p} S \\ \quad \searrow \text{rad } P \nearrow \end{array}$$

$\text{Ker } p$ is max. as S is simple
 $\Rightarrow \text{Ker } p \neq \text{rad } P$.
 $\Rightarrow \text{rad } P$ is maximal
 $\Rightarrow P$ is indecomposable