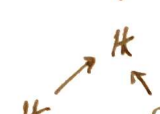


Representasjonsteori av Algebraer - Week 3

Exercises:

• $Q = \left\{ \begin{smallmatrix} i \\ j \end{smallmatrix} \right\}$ ok

• $Q = \left\{ \begin{smallmatrix} \cdot & \cdot \\ \swarrow & \searrow \end{smallmatrix} \right\}$, , ,  
, ,  Are every indec.

• $Q = \left\{ \begin{smallmatrix} \cdot & \cdot \\ \swarrow & \searrow \end{smallmatrix} \right\}$ Every as above... Also have:
 Observe that $\text{End}(e_i) \cong K$, which is local.

• $Q = \left\{ \begin{smallmatrix} \cdot & \cdot \\ \swarrow & \searrow \end{smallmatrix} \right\}$ ∞ many indec.
 Assume $\text{char } K = 0$, then $\rightarrow$ 
 is indec. $\forall e \in K$.

Ex

• R ring, then $M_n(R) \overset{\text{Morita}}{\sim} R$.

$\hookrightarrow R^n$ is a progenerator for $\text{mod } R$, and $M_n(R) \cong \text{End}(R^n)$.

$\hookrightarrow M_n(R)_{M_n(R)} \cong \oplus \text{rows of } M_n(R) \cong (\text{pick a row})^n$, thus a row
 is a progenerator. $\text{End}_{M_n(R)}(\text{one row}) \cong R$.

• S_2 ; group of 2-elements & acts on K^2 by swapping-variables.

$(K \times K)S_2$ is the skew group ring

$(a,b)1 + (a',b')\sigma$ w/ mult s.t. $\sigma(c,d) = (d,c)\sigma$.

$\hookrightarrow (K \times K)S_2 \sim K$

Def. Let A be an Artin ring. A is basic if
 $A_A \cong \bigoplus P_i$, P_i indec. s.t. P_i are pairwise non-isomorphic.

Prop. Let A be an Artin ring. Then $\exists!$ B (up to iso) s.t.
 B basic and $B \sim_{Morita} A$.

Pf. $B \sim_{Morita} A \Leftrightarrow \exists$ progen P s.t. $End_A(P) \cong B$. Decompose

$A_A \cong \bigoplus P_i^{m_i}$, where P_i are indec.

P progen $\Leftrightarrow P \cong \bigoplus P_i^{n_i}$ where $n_i \geq 1$.

B is basic iff every $n_i = 1$. $\square$ $B_B \cong End_A(P) \cong \bigoplus Hom_A(P, P_i)$
 $Hom_A(P, P_i)$ is equiv. thus indec. $\leadsto$ indec.

Obs. A artinian, then A is basic iff $A/\text{rad } A$ is basic.

Thm (Artin-Wedderburn)

A s.s. artinian ring, then $A \cong \prod_i M_n(K_i)$ K_i are skew-fields.

Pf. $A \cong End(A_A) \cong End(\bigoplus S_i^{m_i}) \cong \prod_i Hom(A_A, S_i^{m_i})$
 $\cong \prod_i End(S_i^{m_i}) \cong \prod_i M_n(K_i) \quad \square$

Cor. A f.d. algebra over alg. closed field K , then $A \cong \prod_i M_n(K)$.

Pf. Recall Galois-theory! $\square$

Cor. A basic s.s. artinian, then $A \cong \prod_i K_i$, where K_i are skew-fields.

Cor. A basic s.s. f.d. alg. over alg. closed K , then
 $A \cong \overset{\dim A}{\times} K$.

Cor A f.d. alg. over alg. closed field K .
 P indec. proj.

$\dim_K P / \text{rad } P = \text{multiplicity of } P \text{ as a summand of } A_A.$

Pf. Replacing A by $A / \text{rad } A$ we may assume that A is s.s. For $M_n(K)$ statement holds, both sides are n . $\square$

Cor A f.d. alg. over an alg. closed field K .

Then $A \rightarrow A / \text{rad } A$ splits as a morphism of algebras.

Pf. $A_A \cong \text{End}(A_A) \cong \text{End}(\bigoplus P_i) = (\text{Hom}(P_i, P_j))_{i,j}$ $\leftarrow$ indec. proj.
 $A / \text{rad } A_A = (\{K \text{ if } P_i \cong P_j, 0 \text{ otherwise}\})_{i,j}$ $\leftarrow$ Inclusion $\square$

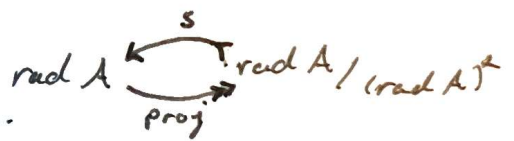
Notation:

Denote $A / \text{rad } A$ as S .
 Note that $S^{\circ P} \otimes_K S$ is still s.s.

Warning: K' extension field of K , then $K' \otimes_K K'$ might not be s.s. It is whenever K' is separable (reference).

The quotient $\text{rad } A \xrightarrow{\text{proj}} \text{rad } A / (\text{rad } A)^2$ is a map of S - S -bimodules by the splitting. As $\text{mod } S^{\circ S}$ is s.s., the quotient proj is in fact split epi.

Denote $\text{rad } A / (\text{rad } A)^2$ by R , and the tensor algebra $T_S R = S \otimes R \otimes S \otimes R \otimes S \dots$



This tensor algebra is the free tensor algebra on the S -bimodule R , i.e. we have the adjunction $\text{Alg}_S(T_S R, V) \cong \text{Mod}_S^S(R, V)$.

We can extend $\iota: R \rightarrow \text{rad } A \rightarrow A$, to get a map $T_S R \rightarrow A$.



Claim: The map is surjective. Show inductively surjective when going to $A / (\text{rad } A)^n$.

$n=1$ S is a summand of $T_S R$, and goes to $A/\text{rad } A$ ✓

$n=2$ $A/(\text{rad } A)^2 \cong S \oplus R$ as bimodules of S . ✓

Assume it works for n .

Pick an $a \in A$, then we may find a $t \in T_S R$, s.t.
 $\hat{S}(t) = \bar{a}$, i.e. $\hat{S}(t) - a \in (\text{rad } A)^n$.

$\hat{S}(t) - a = \sum_{i=1}^{\text{finite}} a_i \cdots a_n$, where each $a_i \in \text{rad } A$.

Let $\bar{a}_i \in R$, then $\hat{S}(\sum_{i=1}^{\text{finite}} \bar{a}_i \otimes \cdots \otimes \bar{a}_n) \in \sum_{i=1}^{\text{finite}} (a_i + (\text{rad } A)^n) \cdots (a_n + (\text{rad } A)^n)$
 $= \sum_{i=1}^{\text{finite}} (a_i \cdots a_n + (\text{rad } A)^{n+1})$

Now, $t - \sum_{i=1}^{\text{finite}} \bar{a}_i \otimes \cdots \otimes \bar{a}_n$ is a preimage of a in $A/(\text{rad } A)^{n+1}$. ✓

Observation:

• $\exists n \in \mathbb{N}$ s.t. $R^{\otimes n} \subseteq \text{Ker } \hat{S}$

• $\text{Ker } \hat{S} \subseteq (R \otimes R)$

Thm

A is a f.d. alg. over an alg. closed field K .

$S = A/\text{rad } A$, $R = \text{rad } A / \text{rad}^2 A$

Then $A \cong T_S R / I$ for some ideal I s.t. $(R^{\otimes n}) \subseteq I \subseteq (R^{\otimes 2})$
 for some $n \in \mathbb{N}$.

Thm

A is a basic f.d. alg. over an alg. closed field K .

Then there is a finite quiver Q and an ideal $I \subseteq KQ$
 s.t. $\exists n \in \mathbb{N}$ where $(\text{arrows})^n \subseteq I \subseteq (\text{arrows})^2$ and

$A \cong KQ / I$.

Pf. In the notation of previous thm $S = K \times \cdots \times K$.
 So let each generator for K represent a vertex.

As R is an S - S -bimodule, each summand of R is K , exactly one copy of K acts non-trivially on the left and one copy acts non-trivial on the right. This summand should then be an arrow of the quiver, going from non-trivial on the right to non-trivial on the left.

Observe that $KQ \cong T_S R$ □

Cor A is a f.d. alg. over an alg. closed field K ,
then A is Morita equivalent KQ/I where

$$(\text{arrows})^n \subseteq I \subseteq (\text{arrows})^1 \quad I \text{ is also called admissible}$$

Slight warning: It is difficult to find such a quiver as above.

Two dualities

We let A be a f.d. alg. over K .

Def: $D = \text{Hom}_K(-, K) : (\text{mod } A)^{\text{op}} \rightarrow \text{mod } (A^{\text{op}})$ is the linear dual.

Observation

- $D^2 \cong \text{Id}$
- S simple $\Leftrightarrow DS$ simple
- P projective $\Leftrightarrow DP$ injective

Consequence

$I \mapsto \text{Soc } I$ is the biggest semi-simple submodule,
which is dual to top .

Gives a bijection $\text{indec. inj} / \cong \rightarrow \text{simple} / \cong$

Now for any ring R .

Construction

We have the functor $\text{Hom}_R(-, R) : (\text{mod } R)^{\text{op}} \rightarrow \text{mod } (R^{\text{op}})$

It induces an equivalence $(\text{proj } R)^{\text{op}} \xrightarrow{\sim} \text{proj } (R^{\text{op}})$,
by additivity.

Warning: These two contravariant functors are typically different.

Def $\nu = \mathbb{D} \operatorname{Hom}_A(-, A)$ is called the Nakayama functor.

Observe ν induces an equivalence $\operatorname{proj} A \rightarrow \operatorname{inj} A$.

Exercise:

show that $\nu \simeq - \otimes_A^L \mathbb{D} A$.

Thm There is a natural isomorphism:

$$\mathbb{D} \operatorname{Hom}_A(P, M) \simeq \operatorname{Hom}_A(M, \nu P), \quad M \in \operatorname{mod} A \text{ and } P \in \operatorname{proj} A.$$

Lf

$$\begin{aligned} h: M \otimes_A \operatorname{Hom}(P, A) &\rightarrow \operatorname{Hom}_A(P, M) \\ m \otimes \varphi &\longmapsto m \cdot \varphi(-) \end{aligned}$$

This is a natural isomorphism.

$$\begin{aligned} \mathbb{D} \operatorname{Hom}_A(P, M) &\simeq \mathbb{D}(M \otimes_A \operatorname{Hom}_A(P, A)) \\ &= \operatorname{Hom}_K(M \otimes_A \operatorname{Hom}_A(P, A), K) \simeq \operatorname{Hom}_A(M, \nu P) \quad \square \end{aligned}$$