# Representasjonsteori av Algebraer – Week 3

Exercises:

- $Q = \{\cdot \to \cdot\}$ ok

- $Q = \left\{\begin{tikzcd}[sep=small] & \cdot \arrow[dl] \arrow[dr] & \\ \cdot & & \cdot\end{tikzcd}\right\}$,
$\begin{tikzcd}[sep=small] & 0 & \\ \mathbb{K} \arrow[ur] & & 0 \arrow[ul]\end{tikzcd}$, $\begin{tikzcd}[sep=small] & 0 & \\ 0 \arrow[ur] & & \mathbb{K} \arrow[ul]\end{tikzcd}$, $\begin{tikzcd}[sep=small] & \mathbb{K} & \\ 0 \arrow[ur] & & 0 \arrow[ul]\end{tikzcd}$

$\begin{tikzcd}[sep=small] & \mathbb{K} & \\ \mathbb{K} \arrow[ur] & & 0 \arrow[ul]\end{tikzcd}$, $\begin{tikzcd}[sep=small] & \mathbb{K} & \\ 0 \arrow[ur] & & \mathbb{K} \arrow[ul]\end{tikzcd}$, $\begin{tikzcd}[sep=small] & \mathbb{K} & \\ \mathbb{K} \arrow[ur] & & \mathbb{K} \arrow[ul]\end{tikzcd}$ Are every indec.

- $Q = \left\{\begin{tikzcd}[sep=small] & \cdot \arrow[dl] \arrow[d] \arrow[dr] & \\ \cdot & \cdot & \cdot\end{tikzcd}\right\}$ Every as above... Also have:

$\begin{tikzcd} & \mathbb{K}^2 & \\ \mathbb{K} \arrow[ur, "\binom{1}{0}"] & \mathbb{K} \arrow[u, "\binom{0}{1}"'] & \mathbb{K} \arrow[ul, "\binom{1}{1}"']\end{tikzcd}$ Observe that $\mathrm{End}(\ ) \simeq \mathbb{K}$, which is local.

- $Q = \left\{\begin{tikzcd}[sep=small] & & \cdot \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\ \cdot & \cdot & \cdot & \cdot\end{tikzcd}\right\}$ $\infty$ many indec.

Assume char $\mathbb{K} = 0$, then $\begin{tikzcd} & \mathbb{K} & & \\ \mathbb{K} \arrow[ur, "\binom{1}{1}"] & \mathbb{K} \arrow[u, "\binom{0}{1}"'] & \mathbb{K} \arrow[ul, "\binom{1}{1}"'] & \mathbb{K} \arrow[ull, bend right, "\binom{1}{\lambda}"']\end{tikzcd}$
is indec. $\forall \lambda \in \mathbb{K}$.

Ex

- $R$ ring, then $M_n(R) \overset{\text{Morita}}{\sim} R$.

$\hookrightarrow R^n$ is a progenerator for $\mathrm{mod}^R$, and $M_n(R) \simeq \mathrm{End}(R^n)$.

$\hookrightarrow M_n(R)_{M_n(R)} \simeq \bigoplus$ rows of $M_n(R) \simeq (\text{pick a row})^n$, thus a row is a progenerator. $\mathrm{End}_{M_n(R)}(\text{one row}) \simeq R$.

- $S_2$; group of 2-elements & acts on $\mathbb{K}^2$ by swapping numbers.

$(\mathbb{K} \times \mathbb{K})S_2$ is the skew group ring

$(a,b)1 + (a',b')\sigma$. w/ mult s.t. $\sigma(c,d) = (d,c)\sigma$.

$\hookrightarrow (\mathbb{K} \times \mathbb{K})S_2 \sim \mathbb{K}$

Def. Let $A$ be an Artin ring. $A$ is basic if

$A_A \simeq \oplus P_i$, $P_i$ indec. s.t. $P_i$ are pairwise non-isomorphic.

Prop. Let $A$ be an Artin ring. Then $\exists!\, B$ (up to iso) s.t. $B$ basic and $B \underset{\text{Morita}}{\sim} A$.

Pf. $B \underset{\text{Morita}}{\sim} A \Leftrightarrow \exists$ progen $P$ s.t. $\mathrm{End}_A(P) \simeq B$. Decompose

$A_A \simeq \oplus P_i^{m_i}$, where $P_i$ are indec.

$P$ progen $\Leftrightarrow P \simeq \oplus P_i^{n_i}$ where $n_i \geq 1$.

$B$ is basic iff every $n_i = 1$. $\square$

$B_B \simeq \mathrm{End}_A(P) \simeq \oplus \mathrm{Hom}_A(P, P_i)$
$\mathrm{Hom}_A(P, -)$ is equiv. thus indec. $\rightsquigarrow$ indec.

Obs. $A$ artinian, then $A$ is basic iff $A/\mathrm{rad}\,A$ is basic.

Thm (Artin-Wedderburn)

$A$ s.s. artinian ring, then $A \simeq \times_i M_n(\mathbb{K}_i)$ $\mathbb{K}_i$ are skew-fields.

Pf: $A \simeq \mathrm{End}(A_A) \simeq \mathrm{End}(\oplus_i S_i^{m_i}) \simeq \times_i \mathrm{Hom}(A_A, S_i^{m_i})$

$\simeq \times_i \mathrm{End}(S^{m_i}) \simeq \times_i M_n(\mathbb{K}_i)$ $\square$

Cor. $A$ f.d. algebra over alg. closed field $\mathbb{K}$, then $A \simeq \times_i M_n(\mathbb{K})$.

Pf: Recall Galois-theory! $\square$

Cor. $A$ basic s.s. artinian, then $A \simeq \times_i \mathbb{K}_i$, where $\mathbb{K}_i$ are skew-fields.

Cor. $A$ basic s.s. f.d. alg. over alg. closed $\mathbb{K}$, then $A \simeq \times^{\dim A} \mathbb{K}$.

__Cor__

$A$ f.d. alg. over alg. closed field $\mathbb{K}$.
$P$ indec. proj.

$\dim_{\mathbb{K}} P/\mathrm{rad}\,P$ = multiplicity of $P$ as a summand of $A_A$.

__Pf.__

Replacing $A$ by $A/\mathrm{rad}\,A$ we may assume that $A$ is s.s. For $M_n(\mathbb{K})$ statement holds, both sides are $n$. □

__Cor__

$A$ f.d. alg. over an alg. closed field $\mathbb{K}$.

Then $A \to A/\mathrm{rad}\,A$ splits as a morphism of algebras.

__Pf.:__

indec. proj.

$$A_A \simeq \mathrm{End}(A_A) \simeq \mathrm{End}(\oplus P_i) = (\mathrm{Hom}(P_i, P_j))_{i,j}$$

$$A/\mathrm{rad}\,A_A = (\{\mathbb{K} \text{ if } P_i \simeq P_j,\ 0 \text{ otherwise})_{i,j}$$

Inclusion □

__Notation:__

Denote $A/\mathrm{rad}\,A$ as $S$.
Note that $S^{op} \otimes_{\mathbb{K}} S$ is still s.s.

__Warning:__ $\mathbb{K}'$ extension field of $\mathbb{K}$, then $\mathbb{K}' \otimes_{\mathbb{K}} \mathbb{K}'$ might not be s.s. It is whenever $\mathbb{K}'$ is seperable (reference).

The quotient $\mathrm{rad}\,A \xrightarrow{\text{proj}} \mathrm{rad}\,A/(\mathrm{rad}\,A)^2$ is a map of $S$-$S$-bimodules by the splitting. As $\mathrm{mod}^{S^{op}\otimes S}$ is s.s., the quotient proj is in fact split epi.

Denote $\mathrm{rad}\,A/(\mathrm{rad}A)^2$ by $R$, and the tensor algebra $T_S R = S \oplus R \oplus R \otimes_S R \oplus \ldots$

$$\begin{tikzcd} \mathrm{rad}\,A \arrow[r, bend right, "\text{proj}"'] & \mathrm{rad}\,A/(\mathrm{rad}\,A)^2 \arrow[l, bend right, "s"'] \end{tikzcd}$$

This tensor algebra is the free tensor algebra a the $S$-bimodule $R$, i.e. we have the adjunction $\mathrm{Alg}^S(T_S R, V) \simeq \mathrm{Mod}_S^S(R, V)$.

We can extend $s: R \to \mathrm{rad}\,A \to A$, to get a map $T_S R \to A$.

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & T_S R \arrow[d, dashed] \\ & A \end{tikzcd}$$

Claim: The map is surjective. Show inductively surjective when going to $A/(\mathrm{rad}\,A)^n$.

$\underline{n=1}$ $S$ is a summand of $T_S R$, and goes to $A/\text{rad}\, A$ ✓

$\underline{n=2}$ $A/(\text{rad}\, A)^2 \simeq S \oplus R$ as bimodules of $S$. ✓

Assume it works for $n$.

Pick an $a \in A$, then we may find a $t \in T_S R$, s.t. $\hat{s}(t) = \bar{a}$, i.e. $\hat{s}(t) - a \in (\text{rad}\, A)^n$.

$$\hat{s}(t) - a = \sum_i^{\text{finite}} a_i^1 \cdots a_n^i, \text{ where each } a_j^i \in \text{rad}\, A.$$

Let $\overline{a_j^i} \in R$, then $\hat{s}\left(\sum_i^{\text{finite}} \bar{a}_i^1 \otimes \cdots \otimes \bar{a}_n^i\right) \in \sum^{\text{finite}} \left(a_i^1 + (\text{rad}\, A)^2\right) \cdots \left(a_n^i + (\text{rad}\, A)^2\right)$
$= \sum^{\text{finite}} \left(a_i^1 \cdots a_n^i + (\text{rad}\, A)^{n+1}\right)$

Now, $t - \sum^{\text{finite}} \bar{a}_i^1 \otimes \cdots \otimes \bar{a}_n^i$ is a preimage of $a$ in $A/(\text{rad}\, A)^{n+1}$. ✓

<u>Observation</u>:

- $\exists n \in \mathbb{N}$ s.t. $R^{\otimes n} \subseteq \text{Ker}\, \hat{s}$
- $\text{Ker}\, \hat{s} \subseteq (R \otimes R)$

<u>Thm</u>

$A$ is a f.d. alg. over an alg. closed field $\mathbb{K}$.

$$S = A/\text{rad}\, A, \quad R = \text{rad}\, A / \text{rad}^2 A$$

Then $A \simeq T_S R / I$ for some ideal $I$ s.t. $(R^{\otimes n}) \subseteq I \subseteq (R^{\otimes 2})$ for some $n \in \mathbb{N}$.

<u>Thm</u>

$A$ is a basic f.d. alg. over an alg. closed field $\mathbb{K}$.

Then there is a finite quiver $Q$ and an ideal $I \subseteq \mathbb{K}Q$ s.t. $\exists n \in \mathbb{N}$ where $(\text{arrows})^n \subseteq I \subseteq (\text{arrows})^2$ and

$$A \simeq \mathbb{K}Q.$$

<u>Pf.</u> In the notation of previous thm $S = \mathbb{K} \times \cdots \times \mathbb{K}$. So let each generator for $\mathbb{K}$ represent a vertex.

As $R$ is an $S$-$S$-bimodule, each summand of $R$ is $\mathbb{K}$, exactly one copy of $\mathbb{K}$ acts non-trivially on the left and one copy acts non trivial on the right. This summand should then be an arrow of the quiver, going from non-trivial on the right to non-trivial on the left.

Observe that $\mathbb{K}Q \simeq T_S R$ □

Cor
A is a f.d. alg. over an alg. closed field $\mathbb{K}$,
then A is Morita equivalent $\mathbb{K}Q/I$ where

$(\text{arrows})^n \subseteq I \subseteq (\text{arrows})^2$     $I$ is also called admissable

Slight warning: It is difficult to find such a quiver as above.

Two dualities

We let A be a f.d. alg. over $\mathbb{K}$.

Def:
$D = \mathrm{Hom}_{\mathbb{K}}(-, \mathbb{K}) : (\mathrm{mod}^A)^{op} \to \mathrm{mod}^{(A^{op})}$ is the linear dual.

Observation
- $D^2 \simeq \mathrm{Id}$
- $S$ simple $\Leftrightarrow$ $DS$ simple
- $P$ projective $\Leftrightarrow$ $DP$ injective

Consequence

$I \mapsto \mathrm{Soc}\, I$ is the biggest semi-simple submodule, which is dual to top.

Gives a bijection indec. inj $/\simeq$ $\to$ simple $/\simeq$

Now for any ring R.

Construction

We have the functor $\mathrm{Hom}_R(-, R) : (\mathrm{mod}^R)^{op} \to \mathrm{mod}^{(R^{op})}$

It induces an equivalence $(\mathrm{proj}\, R)^{op} \xrightarrow{\sim} \mathrm{proj}\,(R^{op})$, by additivity.

Warning: Thes two contravariant functors are typically different.

**Def.** $\nu = \mathbb{D}\operatorname{Hom}_A(-, A)$ is called the Nakayama functor.

**Observe** $\nu$ induces an equivalence $\operatorname{proj} A \to \operatorname{inj} A$.

**Exercise:** Show that $\nu \simeq - \otimes_A \mathbb{D}A$.

**Thm.** Ther is a natural isomorphism:

$$\mathbb{D}\operatorname{Hom}_A(P, M) \simeq \operatorname{Hom}_A(M, \nu P), \quad M \in \operatorname{mod}^A \text{ and } P \in \operatorname{proj} A.$$

**Pf.**

$$h : M \otimes_A \operatorname{Hom}(P, A) \to \operatorname{Hom}_A(P, M)$$
$$m \otimes \varphi \longmapsto m \cdot \varphi(-)$$

This is a natural isomorphism.

$$\mathbb{D}\operatorname{Hom}_A(P, M) \simeq \mathbb{D}(M \otimes_A \operatorname{Hom}_A(P, A))$$
$$= \operatorname{Hom}_{\mathbb{K}}(M \otimes_A \operatorname{Hom}_A(P, A), \mathbb{K}) \simeq \operatorname{Hom}_A(M, \nu P) \qquad \square$$